The spectrum of the Neumann-Poincaré operator of the bowtie

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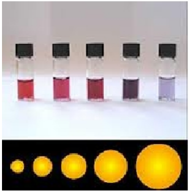
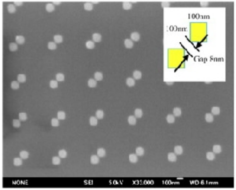
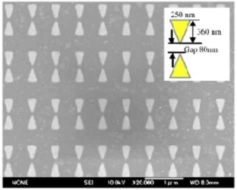
Outline:

1. Introduction/motivation
2. Elliptic corner singularity functions in metamaterials
3. The resonant frequencies of the bowtie with touching wings
4. Bowties with close-to touching wings
5. Conclusion

1. Introduction : Plasmon resonance of metallic nanoparticles







[Gang Bi et al, Optics Comm., 285 (2012) 2472]
Gold particles of diameter << than the wavelength of the exciting radiation

Very small metallic particles exhibit interesting diffractive phenomena, related to resonances : localization and extremely large enhancement of the electromagentic fields in their vicinity

Many potential applications : nanophotonics, nanolithography, near field microscopy, biosensors, cancer therapy

Plasmon resonances may occur in metallic particles if

- the electric permittivity $\varepsilon(\omega)$ inside the particle depends on the frequency of the excitation, and should have a negative real part and a small imaginary part
- the wavelength of the incident excitation $\lambda = 2\pi/\omega$ is much larger than the particle diameter δ

  $$\delta/\lambda \quad = \quad \delta\omega/2\pi \ \prec\prec \ 1$$

  In real life δ is between 10 and 100 nm and $\lambda \sim 650$ nm

The desired resonant frequencies as well as the local fields enhancement may be achieved by tuning the geometry of the nanostructure

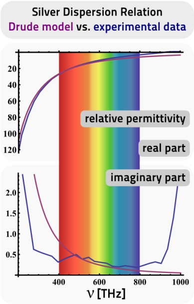
[Mayergoyz-Fredkin-Z Zhang Phys. Rev. B 2005, Grieser Rev. Math. Phys. 14, Hai Zhang]

The Drude model

The Drude model is well accepted for the description of the electric permittivity of metals in the visible light range of frequencies

$$\hat{\epsilon}(\omega) \quad = \quad \varepsilon_\infty - \frac{\omega_P^2}{\omega^2 + i\omega\Gamma}$$

where $\varepsilon_0 > 0,\ \omega_P > 0$ and $\Gamma > 0$ are parameters fitted with experimental data



[P.B. Johnson and R. W. Christy, Phys. Rev. B, 6, 4370-4379 (1972).]

The spectral problem in the electrostatic limit (d=2)

$D \subset \mathbf{R}^2$, bounded C^2 domain with $|D| = 1$

The nanoparticle is centered at a fixed $z \in \mathbf{R}^2$ and occupies

$$D_\delta \quad = \quad z + \delta D$$

A number $\omega \in \mathbf{C}$ is said to be a resonant frequency of the nanoparticle D_δ if there exists a non-trivial solution U to the PDE (in TE polarization):

$$\begin{cases} \Delta U + \omega^2 \varepsilon(x,\omega)\mu_0 U & = & 0 & \text{in } \mathbf{R}^2 \setminus \overline{D_\delta} \cup D_\delta \\ \left[\dfrac{\partial U}{\partial \nu}\right] = [\varepsilon U] & = & 0 & \text{on } \partial D_\delta \\ \lim_{|x| \to \infty} U(x) & = & 0 & \end{cases}$$

where the electric permittivity ε is defined by

$$\varepsilon(x,\omega) \quad = \quad \begin{cases} \varepsilon_0 & \text{for } x \in \mathbf{R}^2 \setminus \overline{D_\delta} \\ \varepsilon_0 \hat{\epsilon}(\omega) & \text{for } x \in D_\delta \end{cases}$$

The change of variable $\tilde{x} = z + x/\delta$ transforms the original spectral problem into

$$\begin{cases} \Delta\tilde{U} + \delta^2\omega^2\varepsilon(x,\omega)\mu_0\tilde{U} & = & 0 & \text{in } \mathbf{R}^2 \setminus \overline{D} \cup D \\ \left[\dfrac{\partial\tilde{U}}{\partial\nu}\right] = \left[\varepsilon\tilde{U}\right] & = & 0 & \text{on } \partial D \\ \lim_{|x|\to\infty} U(x) & = & 0 & \end{cases}$$

where $\tilde{U}(x) = U(\tilde{x})$ and one expects that $\tilde{U}$ converges to a solution of the electrostatic problem

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\varepsilon(x)}\nabla u(x)\right) & = & 0 & \text{in } \mathbf{R}^2 \\ \lim_{|x|\to\infty} u(x) & = & 0 & \end{cases}$$

Integral representation

For convenience, we work in a smooth bounded domain $\Omega \subset \mathbf{R}^2$ and consider a subset $D \subset\subset \Omega$ with Lipschitz boundary

The Poisson kernel associated to D is defined by

$$P(x,y) \quad = \quad G(x,y) + R_x(y), \quad x, y \in \Omega,$$

where $G(x,y)$ is the free space Green function

$$G(x,y) \quad = \quad \frac{1}{2\pi} \ln |x-y|$$

where $R_x(y)$ is the smooth solution to

$$\left\{ \begin{array}{ccc} \Delta_y R_x(y) & = & 0 & y \in \Omega \\ R_x(y) & = & -G(x,y) & y \in \partial\Omega \end{array} \right.$$

We consider the single layer potential $S_D \; : L^2(\partial D) \to L^2(\partial D)$

$$S_D\varphi(x) \quad = \quad \int_{\partial D} P(x,y)\varphi(y)\, ds(y) \quad x \in D \cup (\Omega \setminus \overline{D})$$

$S_D\varphi$ extends as a harmonic function in D and in $\Omega \setminus \overline{D}$, continous in $\overline{\Omega}$

Its normal derivatives satisfy the Plemelj jump conditions

$$\begin{aligned} \frac{\partial S_D\varphi}{\partial \nu}\Big|^{\pm}(x) &= \lim_{t\to 0^+} \nabla S_D\varphi(x \pm t\nu(x)) \cdot \nu(x) \\ &= (\pm\frac{1}{2}I + \mathcal{K}_D^*)\varphi(x) \quad x \in \partial D \end{aligned}$$

where $\mathcal{K}_D^*$ is the Neumann-Poincaré operator

$$\mathcal{K}_D^*\varphi(x) = \int_{\partial D} \frac{\partial P}{\partial \nu_x}(x,y)\varphi(y)\,ds(y)$$

Prop: [Khavinson-Putinar-Shapiro, 2007]

- $\mathcal{K}_D^*$ extends as an operator $H_0^{-1/2}(\partial D) \to H_0^{-1/2}(\partial D)$
- As a consequence of the Calderón identity

  $$\mathcal{K}_D S \quad = \quad S\mathcal{K}_D^*$$

  $\mathcal{K}_D^*$ is self adjoint for the scalar product

  $$<\varphi, \psi>_S \quad = \quad -<\varphi, S_D\psi>_{H^{-1/2},H^{1/2}}$$

- the spectrum of $\mathcal{K}_D^*$ is real and contained in $[-1/2, 1/2]$
- If D is smooth, $\mathcal{K}_D^*$ is compact, so its spectrum consists of eigenvalues that accumulate to 0

The Poincaré variational operator

We define $T_D \ : H^1_0(\Omega) \to H^1_0(\Omega)$ by

$$\forall\ v \in H^1_0(\Omega), \quad \int_\Omega \nabla T_D u \cdot \nabla v \quad = \quad \int_D \nabla u \cdot \nabla v$$

Prop:

- The operator T_D is non-negative, self adjoint, $||T_D|| \leq 1$,
- $Ker(T_D) = \{u \in H^1_0(\Omega),\ u_{|D} = const\}$
- $Ker(I - T_D) = \{u \in H^1_0(\Omega),\ u_{|\Omega \setminus \overline{D}} = 0\}$
- $H^1_0(\Omega) = Ker(T_D) \oplus Ker(I - T_D) \oplus \mathcal{H}$

  where $\mathcal{H}$ is the space of single layer potentials

$$\mathcal{H} \quad = \quad \{u \in H^1_0(\Omega),\ \Delta u = 0 \text{ in } D \cup (\Omega \setminus \overline{D}), \int_{\partial D} \partial_\nu u = 0\}$$

If β is an eigenvalue of T_D with eigenvector u

$$\int_\Omega \beta \nabla u \cdot \nabla v \quad = \quad \int_\Omega \nabla T_D u \cdot \nabla v \;=\; \int_D \nabla u \cdot \nabla v$$

$$\text{i.e. } \int_{\Omega\setminus D} \beta \nabla u \cdot \nabla v \;+\; \int_D (\beta - 1) \nabla u \cdot \nabla v \quad = \quad 0$$

Thus, u is a non-trivial solution to

$$\begin{cases} \operatorname{div}(a(x)\nabla u(x)) = 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{with } a(x) = \begin{cases} 1 & x \in \Omega \setminus \overline{D} \\ k = 1 - 1/\beta & x \in D \end{cases}$$

so that $\varphi = \partial_\nu u|^+ - \partial_\nu u|^-$ satisfies

$$(\lambda I - \mathcal{K}_D^*)\varphi \quad = \quad 0 \qquad \text{with } \lambda = \frac{k+1}{2(k-1)} = 1/2 - \beta$$

In other words, $\sigma(T_D) = 1/2 - \sigma(\mathcal{K}_D^*)$

When D is merely Lipschitz, $\mathcal{K}_D^*$ is no longer compact in general

Thm : [Perfekt-Putinar 2016]

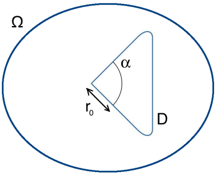
If D is a planar domain with corners, $\sigma(\mathcal{K}_D^*)$ contains essential spectrum and

$$\sigma_{ess}(\mathcal{K}_D^*) \quad = \quad [-(1-\alpha/\pi)/2, (1-\alpha/\pi)/2] \subset\subset [-1/2, 1/2]$$

where α is the most acute angle in D

2. Corner singularity functions

Assume that D is as in the figure



Consider the transmission problem

$$\begin{cases} -\mathrm{div}(a(x)\nabla u(x)) = 0 & \text{in } \Omega \\ u(x) = f & \text{on } \partial\Omega \end{cases} \quad \text{with } a(x) = \begin{cases} 1 & x \in \Omega \setminus \overline{D} \\ k > 0 & x \in D \end{cases}$$

Prop: [Kondratiev, Grisvard, Dauge-Costabel,. . .]

$u(x) = u_{reg} + u_{sing}$ with

$$\begin{cases} u_{reg} \in H^2(\Omega) \\ u_{sing}(x) = Cr^{\eta}\varphi(\theta), \quad 0 < r < r_0, \quad 0 \leq \theta < 2\pi \end{cases}$$

where θ is a smooth function in each sector

$\eta \in (0, 1]$ is determined by α and k (the geometry and the contrast)

What happens when D is filled with a negative index material ?

Seek u_{sing} as a solution to $\mathrm{div}(a\nabla u) = 0$ in the whole plane, with

$$a(x) \quad = \quad a(\theta) \; = \begin{cases} k < 0 & |\theta| < \alpha/2 \\ 1 & \text{otherwise} \end{cases}$$

which has the form $u_{sing} = r^\eta \varphi(\theta)$ with $0 < \eta < 1$

$$\varphi(\theta) \quad = \quad \begin{cases} a_1 \cos(\eta\theta) + b_1 \sin(\eta\theta) & |\theta| < \alpha/2 \\ a_2 \cos(\eta\theta) + b_2 \sin(\eta\theta) & \text{otherwise} \end{cases}$$

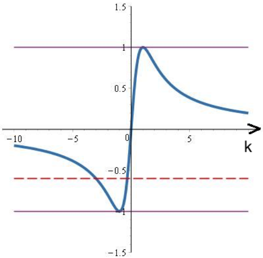
Expressing the transmission conditions $[u] = [a\partial_\theta u] = 0$ on the interfaces $\theta = \pm\alpha/2$ yields a homogeneous linear system for the a_i, b_i's

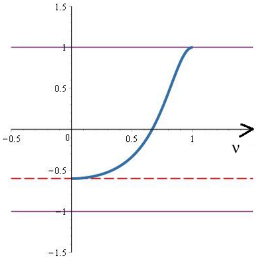
Condition for the existence of non-trivial solutions

$$\frac{2k}{k^2+1} \quad = \quad \frac{\sin(\alpha\eta)\sin((2\pi-\alpha)\eta)}{1-\cos(\alpha\eta)\cos((2\pi-\alpha)\eta)}$$

$$\frac{2k}{k^2+1} = \frac{\sin(\alpha\eta)\sin((2\pi-\alpha)\eta)}{1-\cos(\alpha\eta)\cos((2\pi-\alpha)\eta)}$$

Picture when $\alpha = \Pi/3$:





$$\lambda_\pm = \frac{(k_\pm+1)}{2(k_\pm-1)} = \pm(1-\alpha/\pi)/2$$

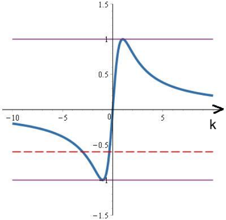
When $k_+ < k < k_- < 0$, one may seek more singular functions in the form

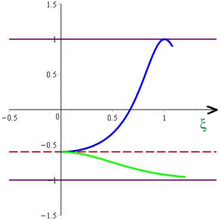
$$u_{sing} \quad = \quad r^{i\xi}\varphi(\theta) \quad \text{with } \xi \in \mathbf{R}$$

for which $\varphi(\theta) \;=\; a_i \cosh(\xi\theta) + b_i \sinh(\xi\theta)$ in each sector

Condition for the existence of non-trivial solutions:

$$\frac{2k}{k^2+1} \quad = \quad \frac{\sinh(\alpha\xi)\sinh((2\pi-\alpha)\xi)}{1-\cosh(\alpha\xi)\cosh((2\pi-\alpha)\xi)}$$



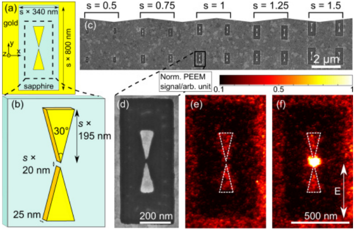


see also [Daugé-Teixier, Bonnet-Chesnel,Bonnet-Chesnel-Clayes]

3. The resonant spectrum of the bowtie

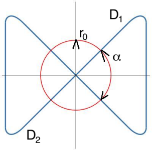
Bowtie nano-antennas are extensively studied in the physics literature, as they can produce a remarkably large enhancement of the electrical field near their corners, and particularly in their central neck

(cf the talk of KiHyun Yun)



(E. Lorek et al, Optics Express Vol. 23, Issue 24, pp. 31460-31471 (2015))

Let D be a bowtie antenna, contained in a set $\Omega \subset \mathbf{R}^2$



Strictly speaking, the bowtie is not a Lipschitz domain : the definition of the Neumann-Poincaré operator may require caution

However, defining a Poincaré variational operator is straightforward

$$\begin{aligned} T_D \ : H_0^1(\Omega) \quad &\longrightarrow \quad H_0^1(\Omega) \\ \forall\, v \in H_0^1(\Omega), \quad \int_\Omega \nabla T_D u \cdot \nabla v \quad &= \quad \int_D \nabla u \cdot \nabla v \end{aligned}$$

The resonant frequencies are related to $\sigma(T_D)$ as (generalized) eigenfunctions of T_D satisfy (in $\mathcal{D}'$)

$$T_D(u) \;=\; \beta u \qquad \Leftrightarrow \qquad \operatorname{div}(a \nabla u) \;=\; 0$$

where
$$a \;=\; \begin{cases} 1 & \text{in } \Omega \setminus D \\ 1 - 1/\beta & \text{in } D \end{cases}$$

Thm : The essential spectrum of the bowtie antenna occupies the whole interval of admissible values

$$\sigma_{ess}(T_D) \quad = \quad [0,1]$$

We prove this result using singular Weyl sequences :

$\beta \in \sigma_{ess}(T_D)$ if and only if there exists a sequence $(u_\varepsilon) \subset H^1_0(\Omega)$ such that

$$\left\{ \begin{array}{lcll} ||u_\varepsilon||_{H^1} & = & 1 & \\ (\beta I - T_D)u_\varepsilon & \rightarrow & 0 & \text{strongly in } H^1 \\ u_\varepsilon & \rightarrow & 0 & \text{weakly in } H^1 \end{array} \right.$$

Proof :

1. The corner singularity functions associated to the central neck of D are easily determined :

Assume that $\beta \in (0,1), \beta \neq 1/2$. Set $k = 1 - 1/\beta$ and

$$a(x) = a(\theta) = \begin{cases} k & \text{if } |\theta| < \alpha/2 \quad \text{and} \quad |\pi - \theta| < \alpha/2 \\ 1 & \text{otherwise} \end{cases}$$

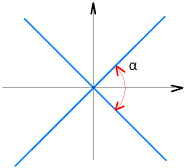
Then there exists a solution u to $\operatorname{div}(a\nabla u) = 0$ in $\mathbf{R}^2$, of the form

$$u(r,\theta) = Re(r^{i\xi})\varphi(\theta), \quad r > 0, \quad 0 \leq \theta < 2\pi$$

for some $\xi > 0$, where

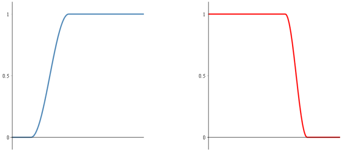
$$\varphi(\theta) = a_i \cosh(\xi\theta) + b_i \sinh(\xi\theta)$$

in each angular sector

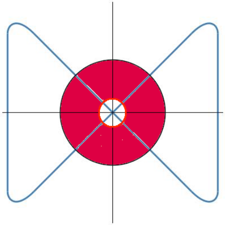


However, the function u is not in H^1_{loc}, as $\nabla u = O(r^{-1})$ near the corner

Let $\varepsilon > 0$ and χ_1, χ_2 be 2 smooth cut-off functions



and define $$u_\varepsilon(x) \; = \; s_\varepsilon \, \chi_1(\tfrac{r}{\varepsilon})\chi_2(r)u(x) \in H^1_0(\Omega)$$



2. We choose s_ε so that $||u_\varepsilon||_{H^1} = ||s_\varepsilon \chi_1(r/\varepsilon)\chi_2 u||_{H^1} = 1$

$$s_\varepsilon^2\left(\int_{\varepsilon<r<2\varepsilon}|u\nabla\chi_1^\varepsilon + \chi_1^\varepsilon\nabla u|^2 + \int_{2\varepsilon<r<r_0/2}|\nabla u|^2 + \int_{\frac{r_0}{2}<r<r_0}|u\nabla\chi_2 + \chi_2\nabla u|^2\right) \quad = \quad 1$$

One can estimate the first term to be bounded by

$$\begin{aligned} \int_\varepsilon^{2\varepsilon}\left[\frac{[\chi_1']^2}{\varepsilon^2} r^{2i\xi}\phi(\theta)^2 \; + \; \frac{\chi_1^2}{r^2} r^{2i\xi}(\varphi(\theta)^2 + [\varphi'(\theta)]^2)\right] r\,dr\,d\theta \\ \leq \quad C\,(O(1) + \ln(2\varepsilon) - \ln(\varepsilon)) \; = \; O(1) \end{aligned}$$

Since $u \notin H^1_{loc}$, the second term tends to $+\infty$

The 3rd term is independent of ε and thus is $O(1)$

It follows that $\quad s_\varepsilon \rightarrow 0 \quad$ and thus that $\quad u_\varepsilon \rightharpoonup 0 \quad$ weakly in H^1

3. We show that $(\beta I - T_D)u_\varepsilon \to 0$ in $H_0^1(\Omega)$

Let $v \in H_0^1(\Omega)$ and compute

$$
\begin{aligned}
&\left| \int_\Omega \nabla(\beta I - T)u_\varepsilon \cdot \nabla v \right| \\
&= \quad \left| \beta \int_{\Omega \setminus D} \nabla u_\varepsilon \cdot \nabla v + (\beta - 1) \int_D \cdot \nabla v \right| = \left| \int_\Omega a \nabla u_\varepsilon \cdot \nabla v \right| \\
&\leq \quad s_\varepsilon \left| \int_\Omega a \nabla(\chi_1(\frac{r}{\varepsilon})\chi_2 u) \cdot \nabla v \right| \\
&\leq \quad s_\varepsilon \left| \int_{B_{2r_0} \setminus B_\varepsilon} a \nabla u \nabla(\chi_1(\frac{r}{\varepsilon})\chi_2 v) - a u \nabla(\chi_1(\frac{r}{\varepsilon})\chi_2) \cdot \nabla v \; - \; \int_{B_{2r_0} \setminus B_{r_0}} a v \nabla \chi_2 \cdot \nabla u \right| \\
&\qquad + s_\varepsilon \left| \int_{B_{2\varepsilon} \setminus B_\varepsilon} a v \nabla \chi_1(\frac{r}{\varepsilon}) \cdot \nabla u \right|
\end{aligned}
$$

The first term above is $O(s_\varepsilon)$

To estimate the term in red

$$\begin{aligned}\int_{B_{2\varepsilon}\setminus B_\varepsilon} av\nabla\chi_1(\frac{r}{\varepsilon})\cdot\nabla u &= \int_{B_{2\varepsilon}\setminus B_\varepsilon} a(v-\overline{v})\nabla\chi_1(\frac{r}{\varepsilon})\cdot\nabla u \\ &\quad + \overline{v}\int_0^{2\pi} a(\theta)\varphi(\theta)d\theta \ \int_\varepsilon^{2\varepsilon} i\xi r^{i\xi-1}\frac{\chi_1'}{\varepsilon} r\,dr\end{aligned}$$

The last term vanishes as φ satisfies $(a\varphi')' - \xi^2 a\varphi = 0$

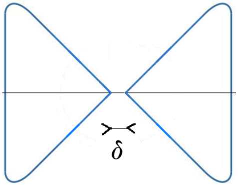
In the first integral, gain a power $\varepsilon^{1/2}$ with the Poincaré-Wirtinger inequality

$$\int_{B_{2\varepsilon}} |v-\overline{v}|^2 \quad \leq \quad C\varepsilon\int_{B_{2\varepsilon}} |\nabla v|^2$$

It follows that $(\beta I - T_D)u_\varepsilon \to 0$ strongly in $H^1(\Omega)$

Conclusion : u_ε is a singular Weyl sequence, and $\beta\in\sigma_{ess}(T_D)$

4. The spectrum of the bowtie with close-to-touching wings



In the case of a bowtie D_δ whose wings are separated by a distance $\delta > 0$, the situation is qualitatively different :

- In that case $\sigma_{ess}(T_{D_\delta}) = [\alpha/\pi, 1-\alpha/\pi] \subsetneq (-1/2, 1/2)$ independently of δ
- When $k > 0$, the regularity of the associated field u_δ also changes qualitatively

$$\begin{cases} u_\delta = r^\eta \varphi(\theta), & \eta \geq 2/3 \quad \forall \alpha, k \\ u_0 = r^\eta \varphi(\theta), & \eta > 0 \quad \text{arbitrary small}(\alpha, k) \end{cases}$$

[E.B., M. Vogelius]

Thm : As $\delta \to 0$, $\sigma(T_{D_\delta})$ must contain eigenvalues

$$\sigma(T_{D_\delta}) \quad = \quad \sigma_{ess}(T_{D_\delta}) \cup \{\beta_i^{\pm}, 1 \leq i \leq N\}$$

The proof is based on the following result of spectral analysis [Allaire-Conca]

Thm : Let $T_\delta \; : H \longrightarrow H$ be a sequence of self-adjoint operators defined on a Hilbert space H

Assume that the T_δ's converge pointwise to a limit operator T

$$\forall\, u \in H, \quad ||T_\delta u - Tu|| \quad \to \quad 0$$

Then $\quad \sigma(T) \subset \lim_{\delta \to 0} \sigma(T_\delta)$

Indeed, one easily sees that $\quad T_{D_\delta} \to T_D$ pointwise

A more direct approach

(that hopefully gives insight on what the eigenfunctions may look like)

Let $\beta > 1 - \alpha/\pi$ so that $\beta \notin \sigma_{ess}(T_{D_\delta})$ for any $\delta > 0$, and let

$$u(x) \quad = \quad Re(r^{i\xi})\varphi(\theta)$$

be a generalized eigenfunction for T_D (i.e. when $\delta = 0$)

Set also

$$u_\varepsilon(x) \quad = \quad s_\varepsilon \chi_1(\frac{r}{\varepsilon})\chi_2(r)u(x)$$

The constant s_ε is chosen so that $||u_\varepsilon|| = 1$ (and thus, $s_\varepsilon \to 0$)

The same computations as before, show that the sequence u_ε satisfies

$$\lim_{\varepsilon \to 0} ||(\beta I - T_D)u_\varepsilon||_{H^1} \quad = \quad 0$$

so that in particular

$$\beta \quad = \quad \lim_{\varepsilon \to 0} \frac{\displaystyle\int_D |\nabla u_\varepsilon|^2}{\displaystyle\int_\Omega |\nabla u|^2}$$

Consider now

$$v_{\varepsilon,\delta}(x_1,x_2) \quad = \quad \begin{cases} u_\varepsilon(x_1+\delta/2, x_2) & \text{if } x_1 < -\delta/2 \\ u_\varepsilon(0, x_2) & \text{if } |x_1| < \delta/2 \\ u_\varepsilon(x_1-\delta/2, x_2) & \text{if } x_1 > \delta/2 \end{cases}$$

By construction $v_{\varepsilon,\delta} \in H_0^1(\Omega)$ and one can estimate

$$\begin{aligned} \int_\Omega |\nabla v_{\varepsilon,\delta}|^2 \quad &= \quad \int_\Omega |\nabla u_\varepsilon|^2 \ + \ s_\varepsilon^2 \int_{|x_1|<\delta/2} |\partial_{x_2}\left[\chi_1(x_2/\varepsilon)\chi_2(x_2)u(0,x_2)\right]|^2 \\ &= \quad 1 \ + \ s_\varepsilon^2 O(\delta/\varepsilon) \end{aligned}$$

and choosing $\varepsilon = \delta$, it follows that

$$\left| \beta - \frac{\displaystyle\int_{D_\delta} |\nabla v_{\delta,\delta}|^2}{\displaystyle\int_\Omega |\nabla v_{\delta,\delta}|^2} \right| \quad \leq \quad \frac{C}{|\ln(\delta)|} \ \rightarrow \ 0$$

For δ sufficiently small, the function v_δ has a Rayleigh quotient above the essential spectrum of T_{D_δ}

However, to be admissible for the min-max principle, it should be orthogonal to $Ker(T_{D_\delta} - I) \sim H_0^1(D_\delta)$

Let W_δ denote the orthogonal projection of $v_{\delta,\delta}$ on $H_0^1(D_\delta)$

$$v_{\delta,\delta} \quad = \quad W_\delta + Z_\delta \qquad \int_\Omega \nabla W_\delta \cdot \nabla Z_\delta = 0$$

Construct U_δ as

$$U_\delta(x) \quad = \quad \begin{cases} W_\delta(x_1 - \delta/2, x_2) & \text{if } x_1 < 0 \\ W_\delta(x_1 + \delta/2, x_2) & \text{if } x_1 > 0 \end{cases}$$

Then $U_\delta \in H_0^1(D) = Ker(T_D - I)$

We can estimate

$$\begin{aligned}
(1-\beta)||W_\delta||^2_{H^1} &= (1-\beta)\int_\Omega |\nabla W_\delta|^2 \;=\; (1-\beta)\int_\Omega \nabla W_\delta\cdot\nabla w_\delta \\
&= \int_\Omega \nabla(T_{D_\delta}-\beta I)W_\delta\cdot\nabla w_\delta \;=\; \int_\Omega \nabla(T_{D_\delta}-\beta I)w_\delta\cdot\nabla W_\delta \\
&= \int_\Omega \nabla(T_D-\beta I)u_\delta\cdot\nabla U_\delta \\
&\le ||(T_D-\beta I)u_\delta||_{H^1}\,||W_\delta||_{H^1}
\end{aligned}$$

It follows that $\lim_{\delta\to 0}||W_\delta||_{H^1}=0$

It follows from the decomposition

$$v_{\delta,\delta} = W_\delta + Z_\delta, \qquad Z_\delta \perp Ker(T_{D_\delta} - I)$$

that

$$\left| \beta - \frac{\displaystyle\int_{D_\delta} |\nabla v_{\delta,\delta}|^2}{\displaystyle\int_\Omega |\nabla v_{\delta,\delta}|^2} \right| \quad \sim \quad \left| \beta - \frac{\displaystyle\int_{D_\delta} |\nabla Z_\delta|^2}{\displaystyle\int_\Omega |\nabla Z_\delta|^2} \right| \to 0$$

where $Z_\delta \in Ker(T_{D_\delta} - I)^\perp$

and therefore, T_{D_δ} has at least one eigenvalue above its essential spectrum

Remarks :

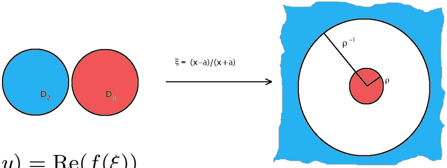
- The symmetric case $\beta \in (0, \alpha/\pi)$ can also be considered, but it is more difficult
- In fact the spectrum contains more and more eigenvalues in the range $[0, \alpha/\pi) \cup 91 - \alpha/\pi, 1]$ as $\delta \to 0$
- [Helsing-Kang-Lim, 2016] contains very nice numerical illustrations of similar phenomena
- The situation is reminiscent of the case of close-to-touching disks

  [Ammari-Ciraolo-Kang-Lee-Milton, McPhedran-Nicorovici-Milton, Bao-Li-Li, EB-Triki]

2 disks centered at $(-1-\delta, 0)$ and $(1+\delta, 0)$

One can transform into 2 concentric disks of radii ρ and $1/\rho$ with the conformal mapping

$$\xi = \frac{z-\alpha}{z+\alpha} \qquad \alpha = \sqrt{\delta(2+\delta)} \qquad \rho = \frac{\alpha-\delta}{\alpha+\delta}$$

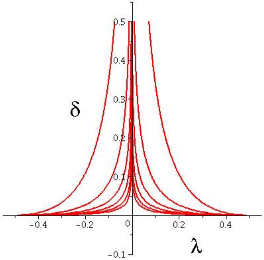


and look for $u(x,y) = \mathrm{Re}(f(\xi))$

The eigenelements of $K^{*,\delta}$ can be explicitly computed

$$\begin{aligned} k_n^- &= -\left(\frac{1+\rho^{2n}}{1-\rho^{2n}}\right) & \lambda-\lambda_n^- &= \lambda-(1/2-n\sqrt{2}\sqrt{\delta}+O(\delta)) \\ k_n^+ &= -\left(\frac{1-\rho^{2n}}{1+\rho^{2n}}\right) & \lambda-\lambda_n^+ &= \lambda-(-1/2+n\sqrt{2}\sqrt{\delta}+O(\delta)) \end{aligned}$$

where $\rho = \dfrac{\sqrt{\delta(2+\delta)}-\delta}{\sqrt{\delta(2+\delta)}+\delta}$



5. Conclusion

- We established a link between the spectral properties of the Neumann-Poincaré operator (or the Poincaré variatonal op.) and the corner singularity functions
- Extension to 3D possible
- The behavior of the associated eigenmodes is also interesting, in view of their properties of localization, concentration of energy
- Are shapes with singularities more interesting for applications ?

  Can that be quantified ?

Save the date !

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