

# Homogenization and the eigenvalues of the Neumann-Poincaré operator

Éric Bonnetier<sup>1</sup>, Charles Dapogny<sup>1</sup> and Faouzi Triki<sup>1</sup>

<sup>1</sup> Laboratoire Jean Kuntzmann, Université Grenoble-Alpes, CNRS, Grenoble, France

8<sup>th</sup> February, 2018

## Foreword: Localized plasmonic resonances

A **localized plasmon resonance** is a phenomenon caused by the interaction between an electromagnetic wave and a **nanoparticle** in a dielectric medium.



The *Lycurgus cup* is encrusted with gold nanoparticles. It looks (left) green when seen in reflection, and (right) red when seen in transmission.

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

- 1 **Localized plasmonic resonances and the Neumann-Poincaré operator**
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
- 2 **Setting and background material**
  - Setting and notations
  - The Poincaré variational principle
- 3 **Asymptotic behavior of the spectrum of the Neumann-Poincaré operator**
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
- 4 **The conductivity equation**
  - Setting
  - Main results
  - The particular case of high-contrast

## Localized plasmonic resonances (II)

- When the nanoparticle is excited by an electromagnetic wave whose frequency is close to a plasmonic resonance,
  - the **absorbing** and **scattering** properties of the particle are strongly enhanced,
  - the electric field blows up in the vicinity of the particle.
- Localized plasmonic resonances occur only in specific situations:
  - The size of the nanoparticle has to be much smaller than the wavelength,
  - The electric permittivity or magnetic permeability of the particle must have **negative** real part, as is the case, e.g. of metallic particles (gold, silver) at optical frequencies.
- The great sensitivity of plasmonic resonances to the local environment of the particle has been used as an ingredient in accurate imaging processes [Ma]:
  - biosensors, gold nanoparticles being harmless for health;
  - spectroscopy devices in biochemistry, to image molecular adsorption on DNA, polymers, etc.

- 1 **Localized plasmonic resonances and the Neumann-Poincaré operator**
  - Localized plasmonic resonances
  - **Mathematical model for plasmonic resonances**
- 2 **Setting and background material**
  - Setting and notations
  - The Poincaré variational principle
- 3 **Asymptotic behavior of the spectrum of the Neumann-Poincaré operator**
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
- 4 **The conductivity equation**
  - Setting
  - Main results
  - The particular case of high-contrast

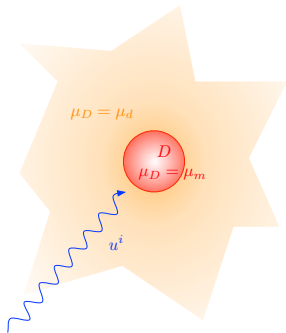
# Mathematical model for plasmonic resonances (I)

The field  $u$  scattered by an incident wave  $u^i$  is solution to the TM **Helmholtz equation**:

$$\left\{ \begin{array}{ll} \operatorname{div}\left(\frac{1}{\mu_D} \nabla u\right) + \omega^2 \varepsilon_D u = 0 & \text{on } D \cup (\mathbb{R}^d \setminus \bar{D}), \\ u^+ = u^- & \text{on } \partial D, \\ \frac{1}{\mu_d} \frac{\partial u^+}{\partial n} = \frac{1}{\mu_m} \frac{\partial u^-}{\partial n} & \text{on } \partial D, \\ u - u^i & \text{satisfies the Sommerfeld} \\ & \text{condition at infinity.} \end{array} \right.$$

When the frequency  $\omega$  is fixed and  $|D| \rightarrow 0$ , a rescaling shows that plasmonic resonances are governed by the existence of non trivial solutions to the **quasi-static equation** [AmMiRuiZha, AmRuiYuZha]:

$$\left\{ \begin{array}{ll} \Delta u = 0 & \text{on } D \cup (\mathbb{R}^d \setminus \bar{D}), \\ u^+ = u^- & \text{on } \partial D, \\ \frac{1}{\mu_d} \frac{\partial u^+}{\partial n} = \frac{1}{\mu_m} \frac{\partial u^-}{\partial n} & \text{on } \partial D, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{array} \right.$$



## Mathematical model for plasmonic resonances (II)

This quasi-static problem is often investigated using **potential theory**;  $u$  is represented as a **single layer potential**  $u = \mathcal{S}_D \phi$ ,  $\phi \in H^{-1/2}(\partial D)$  ( $H_0^{-1/2}(\partial D)$  if  $d = 2$ ):

$$u(x) = \mathcal{S}_D \phi(x) := \int_{\partial D} G(x, y) \phi(y) ds, \text{ where}$$

$$G(x, y) = \begin{cases} \frac{1}{2\pi} \log|x - y| & \text{if } d = 2, \\ \frac{|x - y|^{2-d}}{\omega_d(2-d)} & \text{if } d \geq 3, \end{cases} \text{ is the Newtonian potential.}$$

Using the **Plemelj jump relations** on  $\partial D$ :

$$\frac{\partial u^\pm}{\partial n} = \pm \frac{1}{2} \phi + \mathcal{K}_D^* \phi,$$

where  $\mathcal{K}_D^* : H^{-1/2}(\partial D) \rightarrow H^{-1/2}(\partial D)$  is the **Neumann-Poincaré operator** of  $D$ :

$$\mathcal{K}_D^* \phi(x) = \int_{\partial D} \frac{\partial G}{\partial n_x}(x, y) \phi(y) ds(y),$$

the search for plasmonic resonances boils down to the **eigenvalue problem**:

$$\text{Find } \phi \in H_0^{-1/2}(\partial D) \text{ s.t. } \lambda \phi - \mathcal{K}_D^* \phi = 0, \text{ where } \lambda = \frac{1}{2} \frac{\mu_d + \mu_m}{\mu_d - \mu_m}.$$

### Proposition 1.

If  $D$  is of class  $\mathcal{C}^{1,\alpha}$ ,

- The operator  $\mathcal{K}_D^*$  is *compact*.
- The spectrum  $\sigma(\mathcal{K}_D^*)$  is contained in  $(-\frac{1}{2}, \frac{1}{2}]$ . It consists of a discrete sequence with 0 as unique accumulation point.

The Neumann-Poincaré operator is a key tool in the study of many interface problems with various origins; see [Kan] and references therein:

- Detection and imaging of *inhomogeneities* in an ambient medium,
- *Passive cloaking*, and cloaking by *anomalous localized resonances*,
- Analysis of *stress concentration* between close-to-touching inclusions (metallic particles, elastic fibers, etc.).

## Purposes of the present work

1. Investigate the plasmonic resonances of a large **collection**  $D_1, \dots, D_N$  of  $N$  small particles.

⇒ Do interactions between particles create new resonance effects?

2. Investigate the quasi-static limit of the Helmholtz equation,

$$\begin{cases} -\operatorname{div}(A_D \nabla u) = f & \text{in } \mathbb{R}^d, \\ + \text{conditions at infinity} \end{cases},$$

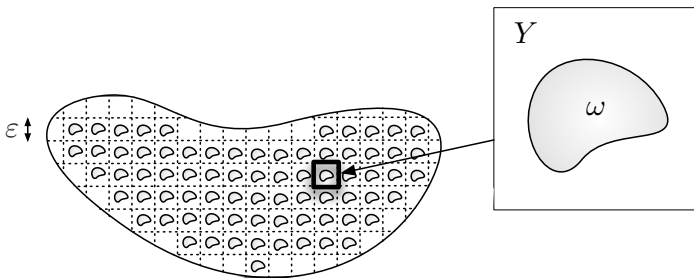
$$\text{where } A_D(x) = \begin{cases} a & \text{if } x \in D_1 \cup \dots \cup D_N, \\ 1 & \text{otherwise,} \end{cases}$$

when the conductivity  $a$  is **negative**, and the number  $N$  of particles **grows to  $\infty$** .

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- 2 **Setting and background material**
  - **Setting and notations**
  - The Poincaré variational principle
  
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## The homogenization setting

Microscopic inclusions with rescaled pattern  $\omega \subset Y := (0, 1)^d$  are periodically distributed in a macroscopic domain  $\Omega \subset \mathbb{R}^d$ .



*Homogenized setting for a periodic distribution of inclusions.*

Working assumptions:

- $\omega$  is smooth and strongly included in  $Y$ :  $\omega \Subset Y$ ;
- $\omega$  and  $Y \setminus \bar{\omega}$  are connected.

## The homogenization setting: notations

- For a point  $x \in \mathbb{R}^d$ :

$$x = \varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_Y ;$$

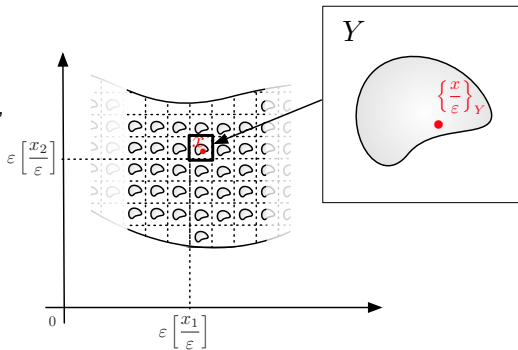
- $\left[ \frac{x}{\varepsilon} \right]_Y \in \mathbb{Z}^d$ : **macroscopic** coordinate,
- $\left\{ \frac{x}{\varepsilon} \right\}_Y \in Y$ : **microscopic** coordinate.

- Indices of the cells inside  $\Omega$ :

$$\Xi_\varepsilon = \left\{ \xi \in \mathbb{Z}^d, \varepsilon(\xi + Y) \Subset \Omega \right\},$$

and corresponding region in  $\Omega$ :

$$\mathcal{O}_\varepsilon = \bigcup_{\xi \in \Xi_\varepsilon} \varepsilon(\xi + Y).$$



- Collection of the portions of  $\varepsilon$ -cells cutting  $\partial\Omega$ :  $\mathcal{B}_\varepsilon := \Omega \setminus \overline{\mathcal{O}_\varepsilon}$ .
- The considered **set of inclusions** is:

$$\omega_\varepsilon = \bigcup_{\xi \in \Xi_\varepsilon} \omega_\varepsilon^\xi, \text{ where } \omega_\varepsilon^\xi := \varepsilon(\xi + \omega).$$

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## The Poincaré variational principle

The study of **plasmonic resonances** involves the **conductivity equation**:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where } A_\varepsilon(x) = \begin{cases} a & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \end{cases} \quad (\mathcal{P}_\varepsilon)$$

the conductivity  $a$  inside  $\omega_\varepsilon$  is in  $\mathbb{C}$ , and  $f \in H^{-1}(\Omega)$  is a source.

A convenient tool is the **Poincaré variational operator**  $T_\varepsilon : H_0^1(\Omega) \rightarrow H_0^1(\Omega)$

[KhaPuSha]:

for  $u \in H_0^1(\Omega)$ ,  $T_\varepsilon u$  is the unique element in  $H_0^1(\Omega)$  s.t.

$$\forall v \in H_0^1(\Omega), \quad \int_{\Omega} \nabla(T_\varepsilon u) \cdot \nabla v \, dx = \int_{\omega_\varepsilon} \nabla u \cdot \nabla v \, dx.$$

Indeed, the conductivity equation  $(\mathcal{P}_\varepsilon)$  is equivalent to:

$$(\lambda \operatorname{Id} - T_\varepsilon)u = \lambda g, \quad \text{where } \lambda = \frac{1}{1-a}, \quad \text{and } f = \nabla g.$$

In particular,  $(\mathcal{P}_\varepsilon)$  is well-posed iff  $\frac{1}{1-a} \notin \sigma(T_\varepsilon)$ , i.e.

$$a \in \mathbb{C} \text{ is a } \text{plasmonic resonance} \text{ of } \omega_\varepsilon \Leftrightarrow \lambda = \frac{1}{1-a} \in \sigma(T_\varepsilon).$$

## The spectrum of $T_\varepsilon$ (I)

- $T_\varepsilon$  is a positive, self-adjoint operator with norm  $\|T_\varepsilon\| \leq 1$ .
- The following **orthogonal decomposition** holds:

$$H_0^1(\Omega) = \text{Ker}(T_\varepsilon) \oplus \mathfrak{h}_\varepsilon \oplus \text{Ker}(\text{Id} - T_\varepsilon),$$

where

- $\text{Ker}(T_\varepsilon) = \{u \in H_0^1(\Omega), u = \text{a cste in each connected component of } \omega_\varepsilon, \}$
- $\text{Ker}(\text{Id} - T_\varepsilon) = \{u \in H_0^1(\Omega), u = 0 \text{ on } \Omega \setminus \overline{\omega_\varepsilon}, \}$
- $\mathfrak{h}_\varepsilon$  ( $\approx$  the space of **single layer potentials**) contains the  $u \in H_0^1(\Omega)$  such that:

$$\begin{cases} \Delta u = 0 & \text{on } \omega_\varepsilon \cup (\Omega \setminus \overline{\omega_\varepsilon}), \\ \int_{\partial\omega_\varepsilon^\xi} \frac{\partial u^+}{\partial n} ds = 0 & \text{for each connected component } \omega_\varepsilon^\xi \text{ of } \omega_\varepsilon. \end{cases}$$

## The spectrum of $T_\varepsilon$ (II)

Key observation: The operator  $R_\varepsilon := T_\varepsilon - \frac{1}{2}\text{Id}$  is related to  $\mathcal{K}_\varepsilon^*$  as:

$$2R_\varepsilon u = (\mathcal{S}_\varepsilon \circ \mathcal{K}_\varepsilon^* \circ \mathcal{S}_\varepsilon^{-1})(u|_{\partial\omega_\varepsilon}), \quad u \in \mathfrak{h}_\varepsilon.$$

### Proposition 2 ([BonTri, KhaPuSha]).

The spectrum of  $T_\varepsilon : \mathfrak{h}_\varepsilon \rightarrow \mathfrak{h}_\varepsilon$  is a *translate of that  $\sigma(\mathcal{K}_\varepsilon^*)$  of the Neumann-Poincaré operator*; it is a discrete sequence of eigenvalues with  $\frac{1}{2}$  as unique accumulation point.

$$0 < \lambda_1^- \leq \lambda_2^- \leq \dots \leq \frac{1}{2}, \quad \text{and} \quad \frac{1}{2} \leq \dots \leq \lambda_2^+ \leq \lambda_1^+ < 1.$$

If  $\{w_i^\pm\}_{i \geq 1}$  are the associated eigenfunctions, the *min-max formulae* hold:

$$\lambda_i^- = \min_{\substack{u \in \mathfrak{h}_\varepsilon \setminus \{0\} \\ u \perp w_1^-, \dots, w_{i-1}^-}} \frac{\int_{\omega_\varepsilon} |\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx} = \max_{\substack{F_i \subset \mathfrak{h}_\varepsilon \\ \dim(F_i) = i-1}} \min_{u \in F_i^\perp \setminus \{0\}} \frac{\int_{\omega_\varepsilon} |\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx},$$

and

$$\lambda_i^+ = \max_{\substack{u \in \mathfrak{h}_\varepsilon \setminus \{0\} \\ u \perp w_1^+, \dots, w_{i-1}^+}} \frac{\int_{\omega_\varepsilon} |\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx} = \min_{\substack{F_i \subset \mathfrak{h}_\varepsilon \\ \dim(F_i) = i-1}} \max_{u \in F_i^\perp \setminus \{0\}} \frac{\int_{\omega_\varepsilon} |\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx}.$$

## Further remarks about $T_\varepsilon$

- For  $u \in H_0^1(\Omega)$ ,  $T_\varepsilon u$  only depends on  $u|_{\omega_\varepsilon} \in H^1(\omega_\varepsilon)$ , modulo a function in

$$C(\omega_\varepsilon) := \left\{ u \in H^1(\omega_\varepsilon), \exists c_\xi \in \mathbb{R}, u = c_\xi \text{ in } \omega_\varepsilon^\xi, \xi \in \Xi_\varepsilon \right\}.$$

- The values of  $T_\varepsilon u$  on  $\Omega \setminus \overline{\omega_\varepsilon}$  may be 'easily recovered' from its values inside  $\omega_\varepsilon$  (since  $T_\varepsilon u$  is harmonic on  $\Omega \setminus \overline{\omega_\varepsilon}$ ).
- The spectrum of the **Neumann-Poincaré operator** can be studied from two complementary points of view:
  - By using **integral equations**, posed on  $\partial\omega_\varepsilon$ ,
  - By **variational methods**, involving the operator  $T_\varepsilon$  (posed on a fixed domain).

## Plasmonic resonances in the homogenization setting

The two concurrent goals pursued in this work rewrite, in the homogenization setting:

1. Analyze the asymptotic behavior of the spectrum  $\sigma(T_\varepsilon)$  in terms of the **limit spectrum**:

$$\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon) = \{ \lambda \in [0, 1], \text{ s.t. } \exists \varepsilon_j \downarrow 0, \lambda_{\varepsilon_j} \in \sigma(T_{\varepsilon_j}), \lambda_{\varepsilon_j} \rightarrow \lambda \}.$$

2. Explore the well-posedness of the **conductivity equation** for the voltage potential,

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases},$$

when the conductivity  $a$  inside the inclusions is **negative**, in the limit  $\varepsilon \rightarrow 0$ .

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - **Uniform bounds on  $\sigma(T_\varepsilon)$**
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## Uniform bounds on the non trivial part of $\sigma(T_\varepsilon)$

One part of the following result was observed in [BuRam]:

### Theorem 3.

There exists  $\varepsilon_0 > 0$  such that, for  $0 < \varepsilon < \varepsilon_0$ ,

$$(\lambda \in \sigma(T_\varepsilon), \lambda \notin \{0, 1\}) \Rightarrow m \leq \lambda \leq M,$$

where  $0 < m < M < 1$  are *explicit* constants:

$$m = \min_{\substack{u \in \widehat{h}_0 \\ u \neq 0}} \frac{\int_{\omega} |\nabla_y u|^2 dy}{\int_Y |\nabla_y u|^2 dy}, \text{ and } M = \max_{\substack{u \in \widehat{h}_0 \\ u \neq 0}} \frac{\int_{\omega} |\nabla_y u|^2 dy}{\int_Y |\nabla_y u|^2 dy},$$

and  $\widehat{h}_0 \subset H^1(Y)/\mathbb{R}$  is the Hilbert space defined by:

$$\widehat{h}_0 = \left\{ u \in H^1(Y)/\mathbb{R}, \Delta_y u = 0 \text{ in } \omega \cup (Y \setminus \bar{\omega}), \text{ and } \int_{\partial\omega} \frac{\partial u^+}{\partial n_y} ds = 0 \right\}.$$

Hint of the proof: Use the **min-max formulae** for the eigenvalues of  $T_\varepsilon : \mathfrak{h}_\varepsilon \rightarrow \mathfrak{h}_\varepsilon$ .

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - **Single cell resonant modes: the cell eigenvalues**
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## How to study the limiting behavior of sequences $\lambda_\varepsilon \in \sigma(T_\varepsilon)$ ?

- $T_\varepsilon$  converges **weakly** to the trivial operator  $|\omega|\text{Id}$ :

$$\forall u \in H_0^1(\Omega), T_\varepsilon u \xrightarrow{\varepsilon \rightarrow 0} |\omega|u, \text{ weakly in } H_0^1(\Omega).$$

- This poor convergence allows to infer nothing about the spectrum  $\sigma(T_\varepsilon)$ .
- As is well-known in **homogenization theory**, **correctors** are needed to obtain a stronger convergence, describing the oscillations of the  $T_\varepsilon u$  at the  $\varepsilon$ -scale.
- These correctors can be used in the study of eigenvalues - see [SanVo, MosVo] - but this approach seems difficult in our context.
- Our work is largely inspired by that of [AlCon] about **Bloch wave homogenization**.  $T_\varepsilon$  is **rescaled** into an operator

$$\mathbb{T}_\varepsilon : L^2(\Omega, H^1(\omega)/\mathbb{R}) \rightarrow L^2(\Omega, H^1(\omega)/\mathbb{R}),$$

which 'does the same' as  $T_\varepsilon$ , but acts on functions  $\phi(x, y)$  depending on both macroscopic and microscopic variables  $x$  and  $y$ .

Definition 1 ([AlCon, CioDamGri]).

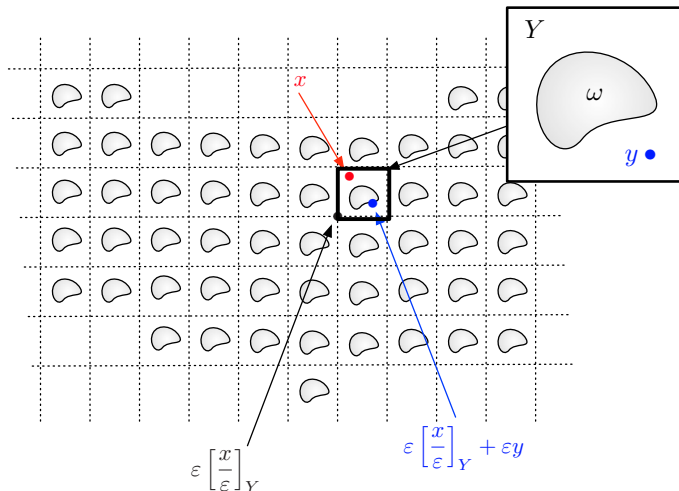
- The **extension** operator  $E_\varepsilon : L^2(\Omega) \rightarrow L^2(\Omega \times Y)$  is defined by:

$$E_\varepsilon u(x, y) = \begin{cases} u(\varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon y) & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

- The **projection** operator  $P_\varepsilon : L^2(\Omega \times Y) \rightarrow L^2(\Omega)$  is defined by:

$$P_\varepsilon \phi(x) = \begin{cases} \int_Y \phi(\varepsilon \left[ \frac{x}{\varepsilon} \right]_Y + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\}_Y) dz & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

# The extension and projection operators $E_\epsilon$ and $P_\epsilon$ (II)



The operator  $E_\epsilon$  rescales the content of each cell to size 1.

## The extension and projection operators $E_\varepsilon$ and $P_\varepsilon$ (III)

- $E_\varepsilon$  and  $P_\varepsilon$  are bounded with norm 1.
- $E_\varepsilon : L^2(\Omega) \rightarrow L^2(\Omega \times Y)$  and  $P_\varepsilon : L^2(\Omega \times Y) \rightarrow L^2(\Omega)$  are **adjoint** operators.
- $E_\varepsilon$  and  $P_\varepsilon$  are '**almost inverse**' from one another:
  - For  $u \in L^2(\Omega)$ ,  $P_\varepsilon E_\varepsilon u(x) = \begin{cases} u(x) & \text{if } x \in \mathcal{O}_\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$
  - For  $\phi \in L^2(\Omega \times Y)$ ,  $E_\varepsilon P_\varepsilon \phi \rightarrow \phi$  strongly in  $L^2(\Omega \times Y)$ , as  $\varepsilon \rightarrow 0$ .

- (**Two-scale convergence**): (see also [Al, Ngue]) Let  $u_\varepsilon$  be a bounded sequence in  $H^1(\Omega)$ ; then, up to a subsequence, there exist  $u_0 \in H^1(\Omega)$  and  $\hat{u} \in L^2(\Omega, H^1_\#(Y))$  such that:

$u_\varepsilon \rightarrow u_0$  weakly in  $H^1(\Omega)$ , and  $E_\varepsilon(\nabla u_\varepsilon) \rightarrow \nabla u_0 + \nabla_y \hat{u}$  weakly in  $L^2(\Omega \times Y)$ .

Intuition: At first order,  $u_\varepsilon$  behaves as:  $u_\varepsilon(x) \approx u_0(x) + \varepsilon \hat{u}(x, \frac{x}{\varepsilon})$ .

## The rescaled operator $\mathbb{T}_\varepsilon$

We define the **rescaled operator**

$$\mathbb{T}_\varepsilon = E_\varepsilon T_\varepsilon P_\varepsilon : L^2(\Omega, H^1(\omega)/\mathbb{R}) \rightarrow L^2(\Omega, H^1(\omega)/\mathbb{R}),$$

which is possible since  $T_\varepsilon$  is defined modulo functions in  $\text{Ker}(T_\varepsilon)$  (i.e. functions that are constant inside each inclusion  $\omega_\varepsilon^\xi$ ).

### Proposition 4.

*The rescaled operator  $\mathbb{T}_\varepsilon$  has the following properties:*

- $\mathbb{T}_\varepsilon$  is self-adjoint.
- $\sigma(\mathbb{T}_\varepsilon) = \sigma(T_\varepsilon) \setminus \{0\}$ .

## Proposition 5.

The operator  $\mathbb{T}_\varepsilon$  **converges pointwise** to a limit  $\mathbb{T}_0$ :

$$\forall \phi \in L^2(\Omega, H^1(\omega)/\mathbb{R}), \quad \mathbb{T}_\varepsilon \phi \xrightarrow{\varepsilon \rightarrow 0} \mathbb{T}_0 \phi, \text{ strongly in } L^2(\Omega, H^1(\omega)/\mathbb{R}).$$

The operator  $\mathbb{T}_0$  is defined by

$$\mathbb{T}_0 \phi(x, y) = Q(\nabla v_0(x) \cdot y + \hat{v}(x, y)), \text{ where}$$

- $Q : L^2(\Omega, H^1(Y)) \rightarrow L^2(\Omega, H^1(\omega)/\mathbb{R})$  is the natural restriction,
- $\hat{v}$  is the unique solution in  $L^2(\Omega, H^1_\#(Y)/\mathbb{R})$  to the equation:

$$-\Delta_y \hat{v}(x, y) = -\operatorname{div}_y(\mathbb{1}_\omega(y) \nabla_y \phi)(x, y) \text{ in } H^1_\#(Y), \text{ a.e. in } x \in \Omega,$$

- $v_0$  is the unique solution in  $H^1_0(\Omega)$  to:

$$-\Delta v_0 = -\operatorname{div} \left( \int_\omega \nabla_y \phi(x, y) dy \right).$$

Hint of proof: This is implied by **two-scale convergence**: the **strong** convergence of sequences  $\mathbb{T}_\varepsilon u$  shows that  $\mathbb{T}_0 u$  'keeps track' of the  $\varepsilon$ -oscillations of the  $\mathbb{T}_\varepsilon u$ .

## Single cell resonances.

This convergence result allows to identify one part of  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$ , corresponding to the **resonance modes** of a **single** inclusion  $\omega \subset Y$ .

### Theorem 6.

The **limit spectrum**  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$  contains the **cell spectrum**, i.e. the spectrum of the operator  $T_0 : H_{\#}^1(Y)/\mathbb{R} \rightarrow H_{\#}^1(Y)/\mathbb{R}$  defined by: for  $u \in H_{\#}^1(Y)/\mathbb{R}$ ,

$$\forall v \in H_{\#}^1(Y)/\mathbb{R}, \quad \int_Y \nabla_y(T_0 u) \cdot \nabla_y v \, dy = \int_{\omega} \nabla_y u \cdot \nabla_y v \, dy.$$

Hint of proof: The **pointwise** convergence  $T_\varepsilon \rightarrow T_0$  allows to infer:

$$\lim_{\varepsilon \rightarrow 0} (\sigma(T_\varepsilon) \setminus \{0\}) = \lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon) \supset \sigma(T_0).$$

In addition, it readily follows from the definition of  $T_0$  that  $\sigma(T_0) \subset \sigma(T_0)$ .



- ① Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- ② Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- ③ Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - **Collective resonances of cells: the Bloch spectrum**
  - Completeness
  
- ④ The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

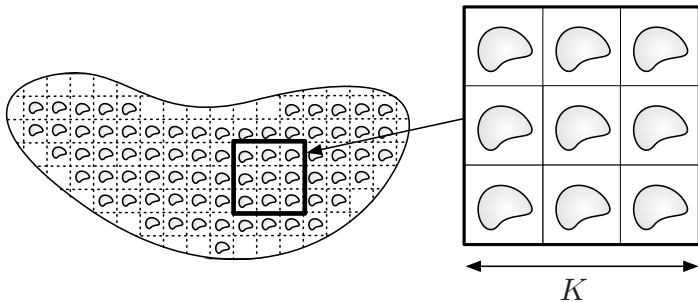
## Rescaling $T_\varepsilon$ over packs of cells (I)

Following [AlCon, Plan], the previous rescaling procedure can be performed over **packs of  $K^d$  cells**,  $K > 1$ .

We define new **extension** and **projection** operators over  $K^d$  cells:

$$E_\varepsilon^K : L^2(\Omega) \rightarrow L^2(\Omega \times KY), \text{ and } P_\varepsilon^K : L^2(\Omega \times KY) \rightarrow L^2(\Omega),$$

which satisfy analogous properties to those of their single-cell counterparts.



Rescaling over a pack of  $K^d$  cells.

## Rescaling $T_\varepsilon$ over packs of cells (II)

- We introduce the collection of  $K^d$  copies of  $\omega$ :

$$\omega^K := \bigcup_{0 \leq j \leq K-1} (j + \omega) \subset KY,$$

and the quotient Hilbert space  $H^K := H^1(\omega^K)/C(\omega^K)$ , where

$$C(\omega^K) := \left\{ u \in H^1(\omega^K), \exists c_j \in \mathbb{R}, u \equiv c_j \text{ on } (j + \omega), 0 \leq j \leq K-1 \right\}.$$

- The **rescaled operator**  $\mathbb{T}_\varepsilon^K : L^2(\Omega, H^K) \rightarrow L^2(\Omega, H^K)$  is now  $\mathbb{T}_\varepsilon^K = E_\varepsilon^K T_\varepsilon P_\varepsilon^K$ .
- Again, we prove the **pointwise convergence** of  $\mathbb{T}_\varepsilon^K$  to a limit operator  $\mathbb{T}_0^K$ :

For all  $\phi \in H^K$ ,  $\mathbb{T}_\varepsilon^K \phi \xrightarrow{\varepsilon \rightarrow 0} \mathbb{T}_0^K \phi$ , strongly in  $H^K$ .

- $\sigma(\mathbb{T}_0^K)$  contains the spectrum of the operator  $T_0^K : H_\#^1(KY)/\mathbb{R} \rightarrow H_\#^1(KY)/\mathbb{R}$  defined by:

$$\forall v \in H^K, \int_{KY} \nabla_y (T_0^K u) \cdot \nabla_y v \, dy = \int_{\omega^K} \nabla_y u \cdot \nabla_y v \, dy.$$

## The Bloch spectrum.

The spectrum  $\sigma(T_0^K)$  is analyzed using a discrete **Bloch decomposition** [AguiCon]:

### Theorem 7.

Let  $u$  in  $L^2_{\#}(KY)$ . Then, there exist  $K^d$  complex-valued functions  $u_j(y) \in L^2_{\#}(Y)$ ,  $j = (j_1, \dots, j_d)$ ,  $j_1, \dots, j_d = 0, \dots, K-1$ , such that:

$$u(z) = \sum_{0 \leq j \leq K-1} u_j(z) e^{\frac{2i\pi j}{K} \cdot z}, \text{ a.e. } z \in KY;$$

The  $u_j$  are unique and are given by:

$$u_j(y) = \sum_{0 \leq j' \leq K-1} u(y + j') e^{-2i\pi \frac{j}{K} \cdot (y + j')}, \text{ a.e. } y \in Y.$$

Furthermore, the **Parseval identity** holds:

$$\forall u, v \in L^2_{\#}(KY), \quad \frac{1}{K^d} \int_{KY} u(z) \overline{v(z)} dx = \sum_{0 \leq j \leq K-1} \int_Y u_j(y) \overline{v_j(y)} dy.$$

## The Bloch spectrum.

Bloch decomposition behaves well with functions  $u \in H^1(\omega^K)$ , and diagonalizes operators with  $Y$ -periodic coefficients. Hence,

$$\sigma(T_0^K) = \bigcup_{0 \leq j \leq K-1} \sigma(T_{\eta_j}), \text{ for } \eta_j = \frac{j}{K},$$

and where the operators  $T_\eta$  are defined by:

- For  $\eta \neq 0$ ,  $T_\eta : H_\#^1(Y) \rightarrow H_\#^1(Y)$  is given by:

$$\forall v \in H_\#^1(Y), \int_Y (\nabla_y(T_\eta u) + 2i\pi\eta(T_\eta u)) \cdot \overline{(\nabla_y v + 2i\pi\eta v)} dy = \int_\omega (\nabla_y(T_\eta u) + 2i\pi\eta u) \cdot \overline{(\nabla_y v + 2i\pi\eta v)} dy.$$

- $T_0 : H_\#^1(Y)/\mathbb{R} \rightarrow H_\#^1(Y)/\mathbb{R}$  is the the same as in the case of a single cell:

$$\forall v \in H_\#^1(Y)/\mathbb{R}, \int_Y \nabla_y(T_0 u) \cdot \overline{\nabla_y v} dy = \int_\omega \nabla_y u \cdot \overline{\nabla_y v} dy.$$

## Theorem 8.

The spectrum  $\sigma(T_\eta)$  is composed of a discrete sequence of real eigenvalues:

$$0 < \lambda_1^-(\eta) \leq \lambda_2^-(\eta) \leq \dots \leq \frac{1}{2} \leq \dots \leq \lambda_2^+(\eta) \leq \lambda_1^+(\eta) \leq 1.$$

Moreover, for any  $i = 1, \dots$ , the mapping  $\bar{Y} \ni \eta \mapsto \lambda_i^\pm(\eta)$  is *Lipschitz continuous*.

Hint of proof:

- This rests on an adapted, **quasi-periodic** version of the Poincaré variational principle, which is tightly connected to the (compact) **quasi-periodic** version of the Neumann-Poincaré operator.
- The Lipschitz continuity of the mappings  $\bar{Y} \ni \eta \mapsto \lambda_i^\pm(\eta)$  relies on the (adapted) min-max formulae, and on an argument of [Ge].

□

## Theorem 9.

The limit spectrum  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$  contains the **Bloch spectrum**  $\sigma_{\text{Bloch}}$  defined by

$$\sigma_{\text{Bloch}} = \bigcup_{i=1}^{\infty} \left[ \min_{\eta \in [0,1]^d} \lambda_i^-(\eta), \max_{\eta \in [0,1]^d} \lambda_i^-(\eta) \right] \cup \bigcup_{i=1}^{\infty} \left[ \min_{\eta \in [0,1]^d} \lambda_i^+(\eta), \max_{\eta \in [0,1]^d} \lambda_i^+(\eta) \right].$$

Hint of proof:

- For every  $K \geq 1$ , the pointwise convergence  $\mathbb{T}_\varepsilon^K \rightarrow \mathbb{T}_0^K$  shows that:

$$\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon) \supset \sigma(\mathbb{T}_0^K) \supset \sigma(T_0^K) = \bigcup_{0 \leq j \leq K-1} \sigma(T_{\eta_j}).$$

- Hence, the limit spectrum contains

$$\bigcup_{i=1}^{\infty} \{ \lambda_i^\pm(\eta_j) \}_{0 \leq j \leq K-1}^{K \geq 1},$$

which gives the desired band structure since the mappings  $\eta \mapsto \lambda_i^\pm(\eta)$  are continuous and  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$  is a closed set.

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - **Completeness**
- 4 The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## The completeness result

The remainder of  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$  gathers the limit behaviors of the eigenvectors of  $T_\varepsilon$  which spend a 'not too small' part of their energy near the macroscopic boundary  $\partial\Omega$ .

### Theorem 10.

The limit spectrum is decomposed as:

$$\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon) = \{0, 1\} \cup \sigma_{\text{Bloch}} \cup \sigma_{\partial\Omega},$$

where the **boundary layer spectrum**  $\sigma_{\partial\Omega}$  is the set of the  $\lambda \in (0, 1)$  such that, for any sequence  $\lambda_\varepsilon \in \sigma(T_\varepsilon)$  with  $\lambda_\varepsilon \rightarrow \lambda$ , and any corresponding (normalized) eigenvector sequence  $u_\varepsilon \in H_0^1(\Omega)$ :

$$\forall s > 0, \lim_{\varepsilon \rightarrow 0} \varepsilon^{-(1-1/d+s)} \|\nabla u_\varepsilon\|_{L^2(\mathcal{U}_\varepsilon)} = \infty,$$

where  $\mathcal{U}_\varepsilon := \{x \in \Omega, d(x, \partial\Omega) < \varepsilon\}$  is the tubular neighborhood of  $\partial\Omega$  with width  $\varepsilon$ .

The difficulty to characterize more precisely  $\sigma_{\partial\Omega}$  reveals **very strong interactions** between the macroscopic boundary  $\Omega$  and the inclusions; see [CasZua, MosVo].

- 1 Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- 2 Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- 3 Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- 4 The conductivity equation
  - **Setting**
  - Main results
  - The particular case of high-contrast

## General setting for the conductivity equation

- We now study the well-posedness and limit behavior of the **conductivity equation**:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where } A_\varepsilon(x) = \begin{cases} a & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{otherwise.} \end{cases} \quad (\mathcal{P}_\varepsilon)$$

The conductivity  $a$  is in  $\mathbb{C}$  and the source  $f$  is in  $H^{-1}(\Omega)$ .

- When  $\operatorname{Im}(a) \neq 0$  or  $a > 0$ , the classical homogenization theory states that  $u_\varepsilon$  converges **weakly in  $H_0^1(\Omega)$**  to the unique solution  $u_*$  of

$$\begin{cases} -\operatorname{div}(A^* \nabla u_*) = f & \text{in } \Omega, \\ u_* = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{P}^*)$$

where the positive definite **homogenized tensor** is defined by:

$$A_{ij}^* = \int_Y A(y) (\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j) dy, \quad \text{where } A(y) = \begin{cases} a & \text{if } y \in \omega, \\ 1 & \text{if } y \in Y \setminus \bar{\omega}. \end{cases}$$

and the **cell functions**  $w_i \in H_{\#}^1(Y)/\mathbb{R}$  solve

$$-\operatorname{div}(A(y)(\nabla w_i + e_i)) = 0 \text{ in } Y, \quad i = 1, \dots, d.$$

- What happens when  $a < 0$ ?

## The formal, homogenized tensor in the case $a < 0$

The cell problems

$$-\operatorname{div}(A(y)(\nabla_y w_i + e_i)) = 0 \text{ in } Y, \quad i = 1, \dots, d.$$

are well-posed provided  $\lambda := \frac{1}{1-a}$  does not belong to the spectrum  $\sigma(T_0)$  of the cell operator  $T_0 : H_{\#}^1(Y)/\mathbb{R} \rightarrow H_{\#}^1(Y)/\mathbb{R}$ :

$$\forall v \in H_{\#}^1(Y)/\mathbb{R}, \quad \int_Y \nabla_y(T_0 u) \cdot \nabla_y v \, dy = \int_{\omega} \nabla_y u \cdot \nabla_y v \, dy.$$

It then makes sense to define the (formal) homogenized tensor

$$A_{ij}^* = \int_Y A(y)(\nabla_y w_i + e_i) \cdot (\nabla_y w_j + e_j) \, dy$$

as soon as  $a \notin \Sigma_{\omega} := \left\{ a \in \mathbb{C}, \frac{1}{1-a} \in \sigma(T_0) \right\}$ .

- ① Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- ② Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- ③ Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- ④ The conductivity equation
  - Setting
  - **Main results**
  - The particular case of high-contrast

## Theorem 11.

Let  $a \in \mathbb{C} \setminus \Sigma_\omega$ ; then,

- If  $u_\varepsilon \in H_0^1(\Omega)$  is a sequence of solutions to  $(\mathcal{P}_\varepsilon)$  such that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C,$$

then up to a subsequence,  $u_\varepsilon$  converges weakly in  $H_0^1(\Omega)$  to a solution of  $(\mathcal{P}^*)$ .

- Conversely, if  $u \in H_0^1(\Omega)$  is one solution to  $(\mathcal{P}^*)$  (if any), then for any sequence  $a_\varepsilon \rightarrow a$ ,  $a_\varepsilon \notin \Sigma_\omega$ , there exists a sequence  $f_\varepsilon \in H^{-1}(\Omega)$  converging pointwise to  $f$  and a sequence  $u_\varepsilon$  of associated voltage potentials, i.e.:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}, \text{ where } A_\varepsilon(x) = \begin{cases} a_\varepsilon & \text{if } x \in \omega_\varepsilon, \\ 1 & \text{otherwise,} \end{cases}$$

such that  $u_\varepsilon \rightarrow u$  weakly in  $H_0^1(\Omega)$ .

This indicates that no 'good' solution to  $(\mathcal{P}^*)$  can be singled out via such a limiting process.

## Proposition 12.

Let  $a \in \mathbb{C} \setminus \{0\}$ . Then  $\frac{1}{1-a}$  belongs to the limit spectrum  $\lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon)$  if and only if there exists  $f \in H^{-1}(\Omega)$  and a sequence  $f_\varepsilon \in H^{-1}(\Omega)$  with  $f_\varepsilon \rightarrow f$  pointwise, such that the solution  $u_\varepsilon \in H_0^1(\Omega)$  of  $(\mathcal{P}_\varepsilon)$  with  $f_\varepsilon$  as a source term satisfies  $\|\nabla u_\varepsilon\|_{L^2(\Omega)^d} \rightarrow +\infty$ .

## Corollary 13.

Let  $a \in \mathbb{C} \setminus \Sigma_\omega$ , and let  $A^*$  be the corresponding homogenized tensor. If

- either there exists  $u \in H_0^1(\Omega)$ ,  $u \neq 0$  such that  $-\operatorname{div}(A^* \nabla u) = 0$ ,
- or there exists  $f \in H^{-1}(\Omega)$  such that  $(\mathcal{P}^*)$  does not have a solution,

then

$$\frac{1}{1-a} \in \lim_{\varepsilon \rightarrow 0} \sigma(T_\varepsilon).$$

- ① Localized plasmonic resonances and the Neumann-Poincaré operator
  - Localized plasmonic resonances
  - Mathematical model for plasmonic resonances
  
- ② Setting and background material
  - Setting and notations
  - The Poincaré variational principle
  
- ③ Asymptotic behavior of the spectrum of the Neumann-Poincaré operator
  - Uniform bounds on  $\sigma(T_\varepsilon)$
  - Single cell resonant modes: the cell eigenvalues
  - Collective resonances of cells: the Bloch spectrum
  - Completeness
  
- ④ The conductivity equation
  - Setting
  - Main results
  - The particular case of high-contrast

## The particular case of high-contrast

The previous material reveals that the conductivity equation  $(\mathcal{P}_\varepsilon)$  is **uniformly well-posed** as  $\varepsilon \rightarrow 0$  when  $a < 0$  is either 'very small' or 'very large'.

### Theorem 14.







*There exists a constant  $0 < \alpha$  such that, if the conductivity  $a$  belongs to  $(-\infty, -1/\alpha) \cup (-\alpha, 0)$ , then*

- (i) For  $0 < \varepsilon$ , the system  $(\mathcal{P}_\varepsilon)$  for  $u_\varepsilon$  is well-posed, i.e. it has a unique solution for any source  $f \in H^{-1}(\Omega)$ , and  $u_\varepsilon$  depends continuously on  $f$ .*
- (ii) The homogenized tensor  $A^*$  is elliptic; in particular,  $(\mathcal{P}^*)$  is well-posed.*
- (iii) For any source  $f \in H^{-1}(\Omega)$ , the unique solution  $u_\varepsilon \in H_0^1(\Omega)$  to  $(\mathcal{P}_\varepsilon)$  converges, weakly in  $H_0^1(\Omega)$ , to the unique solution  $u_*$  of  $(\mathcal{P}^*)$ .*






Thank you !

Thank you for your attention!







## References I

-  [AguiCon] F. Aguirre and C. Conca, *Eigenfrequencies of a tube bundle immersed in a fluid*, Appl. Math. Optim., 18, (1988), pp.1–38.
-  [Al] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. 23, 6, (1992), pp. 1482–1518.
-  [AlCon] G. Allaire and C. Conca, *Bloch wave homogenization and spectral asymptotic analysis*, J. Math. Pures et Appli., 77, (1998), pp.153–208.
-  [AmMiRuiZha] H. Ammari, P. Millien, M. Ruiz and H. Zhang, *Mathematical analysis of plasmonic nanoparticles: the scalar case*, arXiv:1506.00866, (2016).
-  [AmRuiYuZha] H. Ammari, M. Ruiz, S. Yu, and H. Zhang, *Mathematical analysis of plasmonic resonances for nanoparticles: the full Maxwell equations*, arXiv:1511.06817, (2016).
-  [BonDaTri] É. Bonnetier, C. Dapogny and F. Triki, *Homogenization of the eigenvalues of the Neumann-Poincaré operator*, submitted, (2017), arXiv:1702.01798.

## References II

-  [BonTri] E. Bonnetier and F. Triki, *On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2d*, Arch. Rational Mech. Anal., 209, (2013), pp. 541–567.
-  [BuRam] R. Bunoiu and K. Ramdani, *Homogenization of materials with sign changing coefficients*, Commun. Math. Sci., Vol. 14, No. 4 (2016), pp. 1137–1154.
-  [CasZua] C. Castro and E. Zuazua, *Une remarque sur l'analyse asymptotique spectrale en homogénéisation*, C. R. Acad. Sci. Paris, Ser. I 335 pp.99-104 (2002).
-  [CioDamGri] D. Cioranescu, A. Damlamian and G. Griso *The periodic unfolding method in homogenization*, SIAM J. Math. Anal., 40(4), (2008), pp. 1585–1620.
-  [Ge] P. Gérard, *Mesures semi-classiques et ondes de Bloch*, in Séminaire Équations aux Dérivées Partielles 1990-1991, volume 16, Ecole Polytechnique, Palaiseau, (1991).

## References III

-  [Kan] H. Kang, *Layer potential approaches to interface problems*, In *Inverse Problems and Imaging: Panoramas et synthèses*, 44. Société Mathématique de France, (2013).
-  [KhaPuSha] D. Khavinson, M. Putinar and H.S. Shapiro, *On Poincaré's variational problem in potential theory*, *Arch. Rational Mech. Anal.*, 185, (2007), pp. 143–184.
-  [Ma] S. A. Maier, *Plasmonics: Fundamentals and Applications*, Springer, (2007).
-  [MosVo] S. Moskow and M.S. Vogelius, *First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof*, *Proc. Roy. Soc., Edinburgh*, 127, (1997), pp. 1263–1299.
-  [Ngue] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, *SIAM J. Math. Anal.*, 20:3 (1989), pp. 608–623.
-  [Plan] J. Planchard, *Global behaviour of large elastic tube bundles immersed in a fluid*, *Computational Mechanics*, 2, (1987), pp. 105–118.

## References IV



[SanVo] F. Santosa and M.S. Vogelius, *First-Order Corrections to the Homogenized Eigenvalues of a Periodic Composite Medium*, SIAM J. Appl. Math., 53(6), (1993), pp. 1636–1668.