Sub-wavelength acoustic resonators: From super-resolution to metamaterials

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Jeju Island, February 9, 2018

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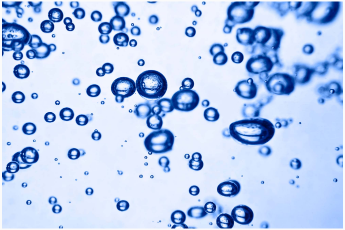
Acoustic bubbles

- Acoustic bubbles can be used to construct acoustic metamaterials.
- There is a high contrast between the mass density of the air in the bubble and the mass density of the surrounding liquid.
- The high contrast leads to a very low resonance frequency making the acoustic bubble a sub-wavelength resonator.
- Acoustic bubbles are very strong monopole scatterers of acoustic waves when excited near the Minnaert resonance frequency.
- Acoustic bubbles can be used to control the propagation of waves whose wavelengths are much larger than the size of the bubbles.



The classical Minnaert resonance

- The natural resonance frequency of a spherical air bubble in water is called the Minnaert resonance frequency:

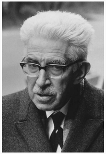
$$\omega_M = \frac{1}{R}\sqrt{\frac{3\kappa_b}{\rho}}.$$

- The Rayleigh-Plesset equation governs the dynamics of the bubble:

$$R\ddot{R} + (3/2)\dot{R}^2 = (1/\rho)(p_0((R_0/R)^{3\gamma} - 1) - p(t))$$

- The Minnaert resonance frequency also appears after linearizing the Rayleigh-Plesset equation:

$$\ddot{R}_\varepsilon(t) + \omega_M^2 R_\varepsilon(t) = -\frac{p(t)}{\rho R_0}.$$



Marcel Minnaert derived the resonance frequency of spherical air bubbles in water in 1933.

Intuition for the Minnaert resonance

- **The acoustic bubble can be viewed as a harmonic oscillator**: For small pressure perturbations, the bubble acts like a harmonic oscillator with angular frequency ω_M.
- **Classic driven harmonic oscillator**:

$$\ddot{x}(t) + \omega^2 x(t) = F(t),$$

where $\omega = \sqrt{\frac{k}{m}}$, with k being the stiffness and m being the mass.

- **Linearized Rayleigh-Plesset equation**:

$$\ddot{R}_\varepsilon(t) + \omega_M^2 R_\varepsilon(t) = -\frac{p(t)}{\rho R_0},$$

where $\omega_M = \sqrt{\frac{3\kappa_b}{R^2\rho}}$, with the bulk modulus of the bubble κ_b acting as the 'stiffness' and the mass density of the liquid ρ acting as the 'mass'.

Applications

- **Acoustic metasurfaces**: Absorb all incoming energy.
- **Phononic crystals**: Prohibit wave propagation.
- **Double negative metamaterials**: Obtain a negative refractive index.
- **Ultrasound constrast agents**: Enhanced reflection of ultrasound waves leading to increased image contrast.

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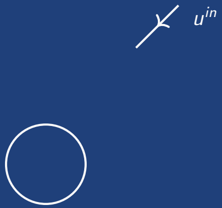
Generalizing the Minnaert resonance

- To construct acoustic metamaterials we need to generalize the notion of the Minnaert resonance for a bubble.
- We require generalized Minnaert resonance formulas for:
  - Arbitrary shaped bubbles
  - Arrays of bubbles (quasi-periodic resonances)
  - Two bubbles located close together (hybridized resonances)
  - A bubble encased in a thin viscoelastic shell (correction to classical resonance formula)
- We are also interested in scattering coefficients which can be used to approximate scattered fields in acoustic bubble problems (and which blow up near Minnaert resonances).

The starting point: A single bubble in an infinite liquid



The starting point: A single bubble in an infinite liquid

$$\begin{cases} \Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D}, \\ \Delta u + k_b^2 u = 0, \quad \text{in} \quad D, \\ u|_+ = u|_-, \quad \text{on} \quad \partial D, \\ \delta \left.\frac{\partial u}{\partial \nu}\right|_+ = \left.\frac{\partial u}{\partial \nu}\right|_-, \quad \text{on} \quad \partial D, \\ u^s := u - u^{in} \text{ satisfies the Sommerfeld R.C. as } |x| \to \infty. \end{cases}$$

- **Wavenumbers**:
$$k = \omega / v = \omega \sqrt{\rho / \kappa}, \qquad k_b = \omega / v_b = \omega \sqrt{\rho_b / \kappa_b}.$$
- **Wavespeeds**: v and v_b are both of order 1.
- **Arbitrary shaped bubble**: Domain D of size order 1 where the boundary $\partial D \in C^{1,s}$ with $0 < s < 1$.
- **High contrast in mass densities**: $\delta^{-1} = (\rho_b / \rho)^{-1} \gg 1$. For an air bubble in water $\delta^{-1} \approx 10^3$.
- **Minnaert resonance**: Smallest ω such that the PDE has a non-trivial solution (with $u^{in} = 0$).

Boundary integral operators

- We introduce boundary integral operators in order to transform the PDE to an equivalent boundary integral problem.
- **Green's function**:
$$G^k(x,y) = -\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}.$$
- **Single layer potential**:
$$\mathcal{S}_D^k[\psi](x) := \int_{\partial D} G^k(x,y)\psi(y)\mathrm{d}\sigma(y).$$
- **Neumann-Poincaré operator**:
$$\mathcal{K}_D^{k,*}[\psi](x) := \int_{\partial D} \frac{\partial}{\partial \nu_x} G^k(x,y)\psi(y)\mathrm{d}\sigma(y).$$

Boundary integral operators

- **Jump relations**:

$$\frac{\partial}{\partial \nu_x} \mathcal{S}_D^k[\psi]\Big|_{\pm}(x) = \Big(\pm \frac{1}{2} I + \mathcal{K}_D^{k,*} \Big)[\psi](x), \quad x \in \partial D.$$

- **Eigenfunction for NP operator**: We define $\psi_0 \in L^2(\partial D)$ such that

$$\Big(-\frac{1}{2} + \mathcal{K}_D^* \Big)\psi_0 = 0, \qquad \int_{\partial D} \psi_0 = 1.$$

- **Capacity**: We define the capacity of the set D by

$$\mathcal{S}_D[\psi_0] = -\mathrm{Cap}_D^{-1} \mathbb{1}_{\partial D}.$$

Boundary integral represenation of the problem

- The solution to the PDE can be represented as

$$u = \begin{cases} u^{in} + \mathcal{S}_D^k[\psi], & \text{in } \mathbb{R}^3 \backslash D, \\ \mathcal{S}_D^{k_b}[\psi_b], & \text{in } D, \end{cases}$$

for $\psi_b, \psi \in L^2(\partial D)$ such that

$$\mathcal{A}_\omega^\delta[\Psi] = F,$$

with

$$\mathcal{A}_\omega^\delta := \mathcal{A}(\omega, \delta) = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^k \\ -\frac{1}{2} + \mathcal{K}_D^{k_b,*} & -\delta(\frac{1}{2} + \mathcal{K}_D^{k,*}) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \psi_b \\ \psi \end{pmatrix}, \ F = \begin{pmatrix} u^{in} \\ \delta \frac{\partial u^{in}}{\partial \nu} \end{pmatrix}.$$

Minnaert resonance for the bubble

- **Resonant frequencies of the bubble**: The complex numbers ω with negative imaginary part and such that the following equation has a non-trivial solution:
$$\mathcal{A}_\omega^\delta[\Psi] = 0.$$
- These ω are the characteristic values of the operator-valued function $\mathcal{A}_\omega^\delta$.
- **Minnaert resonance frequency**: The smallest ω such that the above equation has a non-trivial solution. It is a quasi-static resonance frequency that corresponds to the size of the bubble being much smaller than the wavelength of the incident wave.

Limiting problem

- In the limit as $\rho_b \to 0 \implies \delta \to 0$ we have:

$$\mathcal{A}_\omega^0 = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{k} \\ -\frac{1}{2} + \mathcal{K}_D^{k_b,*} & 0 \end{pmatrix}.$$

- What $\omega \ll 1$ is a characteristic value of this operator?
- Consider $\omega = 0$. In this case we have

$$\mathcal{A}_0^0 = \begin{pmatrix} \mathcal{S}_D & -\mathcal{S}_D \\ -\frac{1}{2} + \mathcal{K}_D^{*} & 0 \end{pmatrix}.$$

- Note that $\Psi_0 = \begin{pmatrix} \psi_0 \\ \psi_0 \end{pmatrix}$ is a non-trivial solution to

$$\mathcal{A}_0^0[\Psi_0] = 0.$$

- So $\omega = 0$ is a characteristic value of the limiting operator $\mathcal{A}_\omega^0$.

Limiting problem

- **Another perspective**: The PDE corresponding to the limiting operator $\mathcal{A}_\omega^0$ when we set $\omega = 0$ is:

$$\begin{cases} \Delta u = 0, \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D}, \\ \Delta u = 0, \quad \text{in} \quad D, \\ u|_+ = u|_-, \quad \text{on} \quad \partial D, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_- = 0, \quad \text{on} \quad \partial D, \\ u = O(|x|^{-1}) \text{ as } |x| \to \infty. \end{cases}$$

- **PDE has a non-trivial solution**: Take u to be constant inside the bubble.
- $\omega = 0$ is a 'Minnaert resonance' for the limiting PDE.

The Minnaert resonance when $0 \neq \delta \ll 1$

- $\omega = \omega(\delta)$ depends on δ continuously.
- **Location of Minnnaert resonance**: Gohberg-Sigal theory tells us that for $0 \neq \delta \ll 1$, the characteristic value ω_M of $\mathcal{A}_\omega^\delta$ will be slightly perturbed away from the characterstic value 0 of the limiting operator $\mathcal{A}_\omega^0$.

Solving $\mathcal{A}_\omega^\delta[\Psi] = 0$

- **Asymptotic expansions**: Expanding $\mathcal{S}_D^k$ and $\mathcal{K}_D^{k,*}$ allows us to write the operator $\mathcal{A}_\omega^\delta$ as $\mathcal{A}_0^0$ plus a small perturbation $\mathcal{B}(\omega,\delta)$:

$$\begin{aligned}\mathcal{A}_\omega^\delta &= \mathcal{A}_0^0 + \mathcal{B}(\omega,\delta)\\ &= \mathcal{A}_0 + \omega\mathcal{A}_{1,0} + \omega^2\mathcal{A}_{2,0} + \omega^3\mathcal{A}_{3,0} + \delta\mathcal{A}_{0,1} + \delta\omega^2\mathcal{A}_{2,1} + O(\omega^4) + O(\delta\omega^3).\end{aligned}$$

- **Simplified problem**: As $\omega, \delta \ll 1$ we can reduce the problem of finding ω such that

$$\mathcal{A}_\omega^\delta[\Psi] = 0,$$

to the equivalent problem of finding ω such that

$$\begin{aligned}0 &= \omega^2\left(\mathcal{A}_{2,0}[\Psi_0],\Phi_0\right) - \omega^3\left(\mathcal{A}_{3,0}[\Psi_0],\Phi_0\right) + \delta\left(\mathcal{A}_{0,1}[\Psi_0],\Phi_0\right)\\ &\quad - \omega^3\left(\mathcal{A}_{2,0}\tilde{\mathcal{A}_0}^{-1}\mathcal{A}_{1,0}[\Psi_0],\Phi_0\right) - \omega\delta\left(\mathcal{A}_{0,1}\tilde{\mathcal{A}_0}^{-1}\mathcal{A}_{1,0}[\Psi_0],\Phi_0\right) + O(\omega^4) + O(\delta^2),\end{aligned}$$

where $\Psi = \Psi_0 + \Psi_1$ with $(\Psi_0, \Psi_1) = 0$.

Minnaert resonance formula

- **Minnaert resonance formula for an arbitrary shaped bubble**:

$$\omega_M(\delta) = \frac{1}{\varepsilon}\left(\sqrt{\frac{v_b^2 Cap(B)}{Vol(B)}}\delta^{\frac{1}{2}} - i\frac{v_b^2 Cap(B)^2}{8\pi v Vol(B)}\delta + O(\delta^{\frac{3}{2}})\right),$$

  where $D = \varepsilon B$ with B being a domain of size order 1.

- **When D is a sphere of radius R we recover the classical Minnaert resonance formula**:

$$\omega_M(\delta) = \frac{1}{R}\sqrt{\frac{3\kappa_b}{\rho}} + O(\delta).$$

Single bubble - Far-field behaviour

- When the observation point x is in the far-field the scattering behaviour of the bubble is:

$$u^s(x) = g(\omega, \delta, D)(1 + O(\omega) + O(\delta) + o(1))u^{in}(y_0)G^k(x, y_0),$$

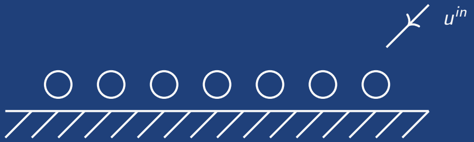
where y_0 is the center of the bubble and g is the scattering coefficient, which is given by

$$g(\omega, \delta, D) = \frac{\mathrm{Cap}(D)}{1 - (\frac{\omega_M}{\omega})^2 + i\gamma},$$

in the regime where $\frac{\omega}{\sqrt{\delta}} = O(1)$. γ is the (radiation) damping coefficient.

- Clearly, scattering will be strongly enhanced when ω is near the Minnaert resonance frequency ω_M.

Perfect energy absorption using a bubble metascreen



Perfect energy absorption using a bubble metascreen

- **Experimental findings**: It has been demonstrated that a periodic layer of bubbles can absorb 50% of incoming energy when the bubbles in the array have appropriate size and position.
- **Perfect absorption**: Placing the layer of bubbles above a perfectly reflecting surface results in a metascreen which can absorb 100% of the incoming energy.
- **Mechanism behind the absorption**: The layer of bubbles act as an equivalent impedance boundary condition with a dependence on the excitation frequency. As the excitation frequency ω is adjusted the reflection coefficient $R(\omega)$ can be made to vary between:
  - $R(\omega) \approx -1$ (Dirichlet b.c.)
  - $R(\omega) \approx 1$ (Neumann b.c.)
  - $R(\omega) \approx 0$ (Perfect absorption - ω near ω_M)

Perfect energy absorption using a bubble metascreen

$$\begin{cases} \Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3_+ \backslash \overline{\Omega}_\varepsilon, \\ \Delta u + k_b^2 u = 0, \quad \text{in} \quad \Omega_\varepsilon, \\ u|_+ = u|_-, \quad \text{on} \quad \partial\Omega_\varepsilon, \\ \delta \dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_-, \quad \text{on} \quad \partial\Omega_\varepsilon, \\ u = 0, \quad \text{on } \partial\mathbb{R}^3_+, \\ u^s := u - u^{in,\alpha} \text{ satisfies an outgoing R.C. as } |x_3| \to \infty. \end{cases}$$

- **Arbitrary shaped bubbles**: $\Omega_\varepsilon = \bigcup_{n \in \mathbb{Z}^2} \varepsilon(D + an)$, where $a > 0$, D is of size order one, and $\varepsilon \ll 1$.
- **Wavenumbers**: k and k_b are both of order 1.
- **Incident wave**: The incident wave $u^{in,\alpha}$ impinges on the metascreen from the upper half-space: $u^{in,\alpha} = u_0 e^{-i\alpha \cdot x}$ where $\alpha_3 > 0$ and $|\alpha| = k$.
- **Quasi-periodic solutions**: Periodic geometry means solutions are Bloch waves.

Green's function for the metascreen

- The quasi-periodic Green's function $G^\alpha(x)$ satisfies:

$$(\Delta + k^2)G^\alpha(x) = \left(\sum_{n\in\mathbb{Z}^2} \delta(x_1 - an_1, x_2 - an_2)e^{-ia(\alpha_1 n_1 + \alpha_2 n_2)}\right)\delta(x_3).$$

- **Green's function for the metascreen**:

$$G_+^\alpha(x,y) = G^\alpha((x_1 - y_1, x_2 - y_2, x_3 - y_3)) - G^\alpha((x_1 - y_1, x_2 - y_2, -x_3 - y_3)).$$

Quasi-periodic boundary integral operators

- **Single layer potential**:

$$\mathcal{S}_+^\alpha[\psi](x) := \int_{\partial D} G_+^\alpha(x,y)\psi(y)\sigma(y),$$

- **Neumann-Poincaré operator**:

$$\mathcal{K}_+^{\alpha,*}[\psi](x) := \int_{\partial D} \frac{\partial G_+^\alpha}{\partial \nu_x}(x,y)\psi(y)\sigma(y).$$

- **Jump relations**:

$$\frac{\partial}{\partial \nu_x}\mathcal{S}_+^\alpha[\psi]\Big|_\pm(x) = \left(\pm\frac{1}{2}I + \mathcal{K}_+^{\alpha,*}\right)[\psi](x), \quad x \in \partial D.$$

- **Eigenfunction for NP operator**: We define $\psi_0 \in L^2(\partial D)$ such that

$$\left(-\frac{1}{2} + \mathcal{K}_+^*\right)\psi_0 = 0, \qquad \int_{\partial D} \psi_0 = 1.$$

- **Capacity**: We define the (quasi-periodic) capacity of the set D by

$$\mathcal{S}_+^\alpha[\psi_0] = -(\mathrm{Cap}_D^+)^{-1}\mathbb{1}_{\partial D}.$$

Rescaled problem

- **Resonant regime**: $\delta \sim \varepsilon^2$. Fix $\mu > 0$ such that $\delta = \mu\varepsilon^2$.
- **Microscopic variabes**: Denote by $\tilde{x} := x/\varepsilon$ and $\tilde{u}(\tilde{x}) := u(\varepsilon x)$.
- **Rescaled problem**:

$$
\begin{cases}
\Delta \tilde{u} + \varepsilon^2 k^2 \tilde{u} = 0, \quad \text{in} \quad \mathbb{R}^3_+ \backslash \overline{\Omega}_1, \\
\Delta \tilde{u} + \varepsilon^2 k_b^2 \tilde{u} = 0, \quad \text{in} \quad \Omega_1, \\
\tilde{u}|_+ = \tilde{u}|_-, \quad \text{on} \quad \partial\Omega_1, \\
\mu\varepsilon^2 \left.\dfrac{\partial \tilde{u}}{\partial \nu}\right|_+ = \left.\dfrac{\partial \tilde{u}}{\partial \nu}\right|_-, \quad \text{on} \quad \partial\Omega_1, \\
\tilde{u} = 0, \quad \text{on } \partial\mathbb{R}^3_+, \\
\tilde{u}^s := \tilde{u} - \tilde{u}^{in,\varepsilon\alpha} \text{ satisfies an outgoing R.C. as } |x_3| \to \infty,
\end{cases}
$$

Boundary integral represenation of the problem

- The solution to the rescaled PDE can be represented as

$$\tilde{u}(x) = \begin{cases} u^{in,\varepsilon\alpha}(x_1, x_2, x_3) - u^{in,\varepsilon\alpha}(x_1, x_2, -x_3) + \mathcal{S}_+^{\varepsilon\alpha}[\psi], & \text{in } \mathbb{R}^3 \backslash \Omega_1, \\ \mathcal{S}_+^{\varepsilon\alpha_b}[\psi_b], & \text{in } \Omega_1, \end{cases}$$

for $\psi_b, \psi \in L^2(\partial D)$ such that

$$\mathcal{A}^\varepsilon[\Psi] = F,$$

with

$$\mathcal{A}^\varepsilon := \mathcal{A}(\varepsilon) = \begin{pmatrix} \mathcal{S}_+^{\varepsilon\alpha_b} & -\mathcal{S}_+^{\varepsilon\alpha} \\ -\frac{1}{2} + \mathcal{K}_+^{\varepsilon\alpha_b,*} & -\mu\varepsilon^2(\frac{1}{2} + \mathcal{K}_+^{\varepsilon\alpha,*}) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} \psi_b \\ \psi \end{pmatrix}, \; F = \begin{pmatrix} u^{in,\varepsilon\alpha}(x_1, x_2, x_3) - u^{in,\varepsilon\alpha}(x_1, x_2, -x_3) \\ \mu\varepsilon^2 \frac{\partial}{\partial\nu}(u^{in,\varepsilon\alpha}(x_1, x_2, x_3) - u^{in,\varepsilon\alpha}(x_1, x_2, -x_3)) \end{pmatrix}.$$

Expansion of the operator $\mathcal{A}^\varepsilon$

- **Asymptotic expansions**: Expanding the boundary integral operators allows us to write the operator $\mathcal{A}^\varepsilon$ as $\mathcal{A}^0$ plus a small perturbation $\mathcal{B}(\varepsilon)$:

$$\mathcal{A}^\varepsilon = \mathcal{A}^0 + \mathcal{B}(\varepsilon).$$

- **Simplified problem**: As $\epsilon \ll 1$ we can simplify and solve the problem of finding $\Psi = (\psi_b, \psi)^T$ such that

$$\mathcal{A}^\varepsilon[\Psi] = F.$$

  Upon doing so we obtain the scattered field.

- **Microscopic scattered field**:

$$\begin{aligned}\tilde{u}^s(x) &= \mathcal{S}_+^{\varepsilon\alpha}[\psi](x)\\ &= 2\mathrm{i}u_0\alpha_3 \int_{\partial D} G_+^{\varepsilon\alpha}(x,y)\Big(\underbrace{\varepsilon(\psi_1^+ - M_1 C_D^+ \psi_0^+)}_{\text{dipole moment}} + \underbrace{g_s(\mu,\varepsilon) C_D^+ \psi_0^+}_{\text{monopole moment}}\Big)(y)\sigma(y) + O(\varepsilon^2).\end{aligned}$$

Approximate solution

- As the periodicity of the lattice is $\varepsilon \ll 2\pi/k = \lambda$, only one mode propagates while the rest are evanescent.
- **Decompose scattered field**: Split scattered field into a propagating mode $\tilde{u}_1$ plus an infinite number of evanescent modes $\tilde{u}_{BL}$:

$$\tilde{u}^s \approx \tilde{u}_1 + \tilde{u}_{BL}.$$

- **Transform back to macroscopic variables**: The approximate solution is

$$\begin{aligned} u(x) &= u^0(x) + u^s(x) + O(\varepsilon^2) \\ &= u^0(x) + \left(u_1(x) + u_{BL}\left(x, \frac{x}{\varepsilon}\right)\right) + O(\varepsilon^2), \end{aligned}$$

where

- u^0 is the background solution in the absence of bubbles.
- u_{BL} is exponentially decaying away from the bubbles.
- u_1 reflects the scattering behaviour in the far-field.

Perfect absorption mechanism

- **Approximate scattered field**:

$$u^s \approx R_\varepsilon(\omega) u^{in,\alpha}(x_1, x_2, -x_3)$$
$$= (-1 + 2i\alpha_3(\varepsilon\beta_{1,\infty} - g_s(\omega,\varepsilon)\beta_{0,\infty}) u^{in,\alpha}(x_1, x_2, -x_3).$$

- **Away from the Minnaert resonance**: The scattered field is of size order ε and the bubbles are essentially invisible.
- **Near the Minnaert resonance**: The monopole component can affect the far-field due to blow-up of the scattering function $g_s(\omega, \varepsilon)$.
- **Reflection coefficient**: The reflection coefficient is of the form

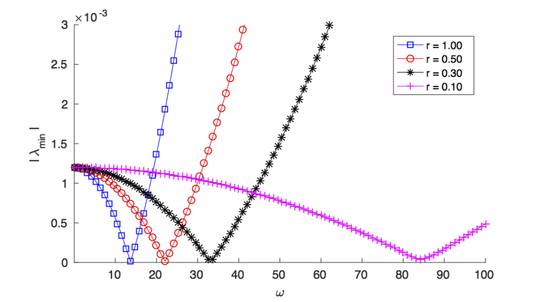
$$R_\varepsilon(\omega) \approx -1 - \frac{2i\varepsilon\eta}{1 - (\omega/\omega_M^+)^2 - i\varepsilon\eta^*}, \quad \eta^* = \eta.$$

  - Away from the Minnaert resonance $R_\varepsilon(\omega) \approx -1 - O(\varepsilon) \approx -1$ (Dirichlet b.c.).
  - At the Minnaert resonance $R_\varepsilon(\omega) \approx -1 - (-2i\varepsilon\eta/(i\varepsilon\eta)) \approx 1$ (Neumann b.c.).

- **Metascreen is tunable**: Ranges between a Dirichlet and a Neumann surface.
  - The damping coefficient $\eta*$ in the denominator neglects thermal and viscous damping. If $\eta*$ were to take account of these damping mechanisms then we should excite the bubbles at a certain frequency near the Minnaert resonance such that

$$R_\varepsilon(\omega) = 0 \quad \text{(Perfect absorption)}$$

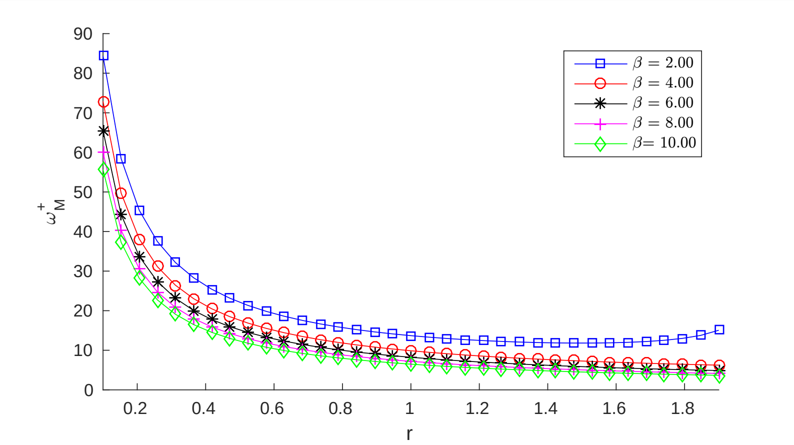
Minnaert resonance with respect to bubble size



r	ω_M^+	ω_c^+	$\frac{2\pi}{k}$
1	13.692	13.674	682
0.5	22.103	22.091	422
0.3	33.183	33.158	281
0.1	84.356	84.324	111

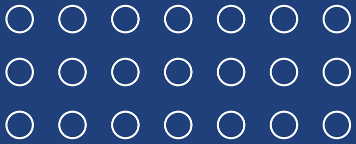
For bubbles much smaller than the wavelength, the quasi-periodic Minnaert resonance formula $\omega_M^+ = (\delta C_D^+/|D|)^{1/2} v_b/R$ is very close to the numerically calculated resonance ω_c^+

Minnaert resonance with respect to distance to geometry



The quasi-periodic Minnaert resonance ω_M^+ of a periodic array of bubbles above a surface, where r is the radius of the bubbles and β is the distance to the surface.

A phononic crystal constructed from bubbles



A phonic crystal constructed from bubbles

- **Prohibiting acoustic wave propagation**: Usually the period of the structure must be of the order of the wavelength to open a bandgap in a phononic crystal. This is impractical in applications.
- **Sub-wavelength bandgap opening**: Acoustic bubbles acting as sub-wavelength resonators can be used to construct phononic crystals in which waves with wavelengths much larger than the size of the bubbles can be prohibited from propagating.
- **Location of sub-wavelength bandgap**: In a high contrast bubble crystal, bandgap opening occurs slightly above the Minnaert resonance frequency of a single bubble.

A phonic crystal constructed from bubbles

$$
\begin{cases}
\Delta u + k^2 u = 0 \quad \text{in} \quad Y \backslash \overline{D}, \\
\Delta u + k_b^2 u = 0 \quad \text{in} \quad D, \\
u_+ = u_- \quad \text{on} \quad \partial D, \\
\delta \dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_-, \quad \text{on} \quad \partial D, \\
e^{-i\alpha\cdot x} u \ \text{ is periodic.}
\end{cases}
$$

- **Wavespeeds**: v and v_b are both of order 1.
- **Arbitrary shaped bubble**: Domain D of size order 1 in the unit cell Y.
- **Quasi-periodic solutions**: Periodic geometry means solutions are Bloch waves.
- **Quasi-periodic Minnaert resonance**: For each α in the Brillouin zone $(-\pi, \pi)^3$ we have a quasi-periodic Minnaert resonance. Plotting these resonances over α in the irreducible Brillouin zone gives us the first band in the band diagram for the crystal.

Boundary integral represenation of the problem

- **Quasi-periodic boundary integral operators**: Introduce $\mathcal{S}_D^{\alpha,k}$ and $(\mathcal{K}_D^{-\alpha,k})^*$ based on the quasi-periodic Green's function $G^{\alpha,k}(x,y)$ given by

$$G^{\alpha,k}(x,y) = \sum_{n\in\mathbb{Z}^3} \frac{e^{i(2\pi n+\alpha)\cdot(x-y)}}{k^2 - |2\pi n+\alpha|^2}.$$

- **Boundary integral formulation**: The quasi-periodic Minnaert resonance frequency for the bubble crystal is the smallest $\omega(\alpha)$ such that the following equation has a non-trivial solution:

$$\mathcal{A}_\omega^\delta[\Psi] = 0,$$

where $\psi_b, \psi \in L^2(\partial D)$ and

$$\mathcal{A}_\omega^\delta := \mathcal{A}(\omega,\delta) = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{\alpha,k} \\ -\frac{1}{2} + \mathcal{K}_D^{k_b,*} & -\delta(\frac{1}{2} + (\mathcal{K}_D^{-\alpha,k})^*) \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \psi_b \\ \psi \end{pmatrix}.$$

- **Limiting case**: $\mathcal{A}_0^0[\Psi] = 0$ has a non-trivial solution:

$$\Psi_0 = \begin{pmatrix} \psi_0 \\ \tilde{\psi}_0 \end{pmatrix},$$

where $\tilde{\psi}_0 = (\mathcal{S}_D^{\alpha,0})^{-1} S_D[\psi_0]$ and $\int_{\partial D} \psi_0 = 1$.

Minnaert resonance for the bubble phononic crystal

- **Asymptotic expansions**: The operator $\mathcal{A}_\omega^\delta$ can be written as $\mathcal{A}_0^0$ plus a small perturbation $\mathcal{B}(\omega, \delta)$:

$$\begin{aligned}\mathcal{A}_\omega^\delta &= \mathcal{A}_0^0 + \mathcal{B}(\omega, \delta) \\ &= \mathcal{A}_0^0 + \omega\mathcal{A}_{1,0} + \omega^2\mathcal{A}_{2,0} + \omega^3\mathcal{A}_{3,0} + \delta\mathcal{A}_{0,1} + \delta\omega^2\mathcal{A}_{2,1} + O(|\omega|^4 + |\delta\omega^3|).\end{aligned}$$

- **Simplify and solve**: Again, as $\omega, \delta \ll 1$, the problem can be simplified and we can solve for ω such that $\mathcal{A}_\omega^\delta[\Psi] = 0$.
- **Quasi-periodic Minnaert resonance frequency**:

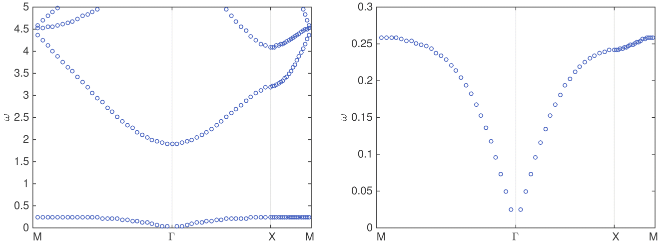
$$\omega_1^\alpha = \omega_{M,\alpha} + O(\delta^{3/2}) = \omega_M \sqrt{\frac{\mathrm{Cap}_{D,\alpha}}{\mathrm{Cap}_D}} + O(\delta^{3/2}),$$

where ω_M is the single bubble Minnaert resonance frequency and $\mathrm{Cap}_{D,\alpha}$ is the quasi-periodic capacity defined as

$$\mathrm{Cap}_{D,\alpha} := -((\mathcal{S}_D^{\alpha,0})^{-1}[\mathbb{1}_{\partial D}], \mathbb{1}_{\partial D}).$$

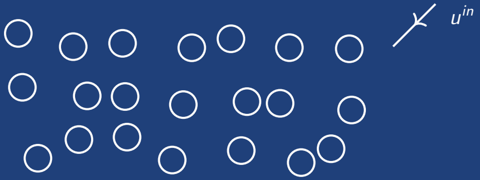
Sub-wavelength bandgap opening

- **Suwavelength bandgap**: For sufficiently high contrast a subwavelength bandgap will open.



The band structure of a two-dimensional periodic array of bubbles of radius $r = 0.05$, with contrast $\delta^{-1} = 5000$.

Sub-wavelength focusing of acoustic waves in bubbly media



Sub-wavelength focusing of acoustic waves in bubbly media

- When we emit acoustic waves from a point source the waves spread out during propagation.
- We would like to focus them back to the original point source.
- **Far-field time-reversal**: Record the acoustic signal using a time reversal mirror in the far-field, time-reverse it, and then re-emit the signal.
- **Diffraction limit**: Evanescent components of the signal never reach the time-reversal mirror and hence the focal spot size of the re-emitted signal is constrained by the diffraction limit of $\lambda/2$.
- **Overcoming the diffraction limit**: If the point source is surrounded by resonant bubbly media it is possible to recover the evanescent contribution and beat the diffraction limit. This phenomenon is called super-focusing or super-resolution in imaging.
- **Approach**: Define a point-scatterer approximation and then use it as the foundation for a numerical investigation of sub-wavelength focusing.

Sub-wavelength focusing of acoustic waves in bubbly media

$$\begin{cases} \Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D^N}, \\ \Delta u + k_b^2 u = 0, \quad \text{in} \quad D^N, \\ u|_+ = u|_-, \quad \text{on} \quad \partial D^N, \\ \delta \left.\dfrac{\partial u}{\partial \nu}\right|_+ = \left.\dfrac{\partial u}{\partial \nu}\right|_-, \quad \text{on} \quad \partial D^N, \\ u^s := u - u^{in} \text{ satisfies the Sommerfeld R.C. as } |x| \to \infty. \end{cases}$$

- **Wavespeeds**: v and v_b are both of order 1.
- **Large number of spherical bubbles**: D^N represents the domain occupied by $N \gg 1$ identical spherical bubbles of radius R which is of order 1:

$$D^N := \cup_{1 \leq j \leq N} D_j^N, \qquad D_j^N = z_j^N + D, \qquad 1 \leq j \leq N.$$

  The bubbles are uniformly distributed in a cube at the origin.
- **Source field**: u^{in} is emitted from a point source at the origin.
- **Gas concentration**: The bubbly medium is very dilute.

Super-focusing of acoustic waves in bubbly media

- **Scattering function for a single spherical bubble**:

$$g_s(\omega) := \frac{1}{\left(\frac{\omega_M}{\omega}\right)^2 - 1 - i\gamma},$$

where $\gamma = \gamma_{rad} + \gamma_{th} + \gamma_{vis}$ is the damping coefficient which represents dissipation due to radiation, thermal, and viscous damping.

- **Approximate total field (point-scatterer approximation)**:

$$u(x) = u^{in}(x) + \sum_{1 \le i \le N} -4\pi R g_s(\omega) G^k(x - x_j) \sum_{1 \le j \le N} (M^{-1})_{ij} u^{\mathrm{in}}(x_j),$$

where M is an $N \times N$ matrix given by

$$M_{ij} = \begin{cases} 1, & i = j, \\ 4\pi R g_s(\omega) G^k(x_i - x_j), & i \neq j. \end{cases}$$

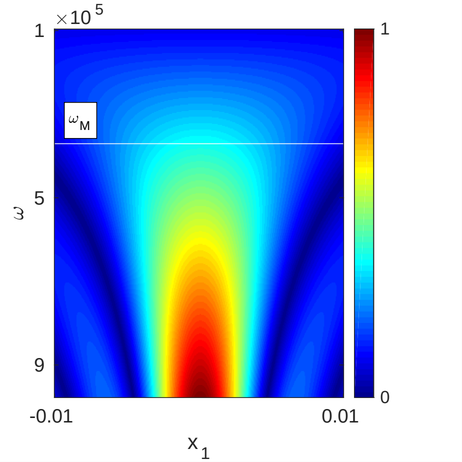
A priori evaluation of focusing performance

- Prior to the performing a time-reversal simulation we would like to have an idea of the optimal frequency range for achieving sub-wavelength focusing.
- **Effective Green's function for the bubbly media**:

$$G_{eff}(x,\omega) := G(x,\omega/v) - \sum_i 4\pi R g_s(\omega) G(x - x_i, \omega/v) \sum_j (M^{-1})_{ij} G(x_j, \omega/v).$$

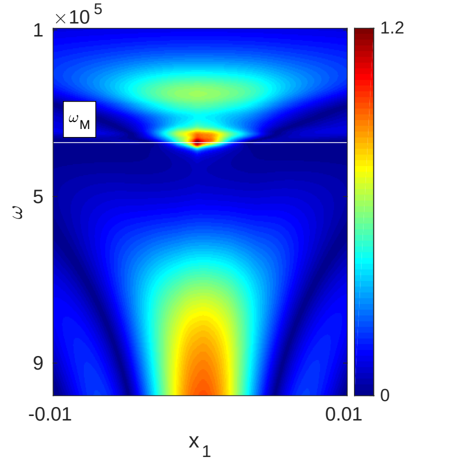
- **Measure of focusing performance**: The imaginary part of the Green's $\text{Im}(G_{eff})$ function is the point spread function. The more point-like $Im(G_{eff})$ is, the sharper the focusing. Additionally, in the time-reversal simulation the focusing performance is measured by the full width at half maximum (FWHM) of the focal spot.
- **Optimal focusing frequency**: We analyze $\text{Im}(G_{eff})$ in the presence of a point source at the origin over a range of frequencies for increasing volume fractions of bubbles to find out where it becomes point-like.

A priori evaluation of focusing performance - Im(G_{eff})



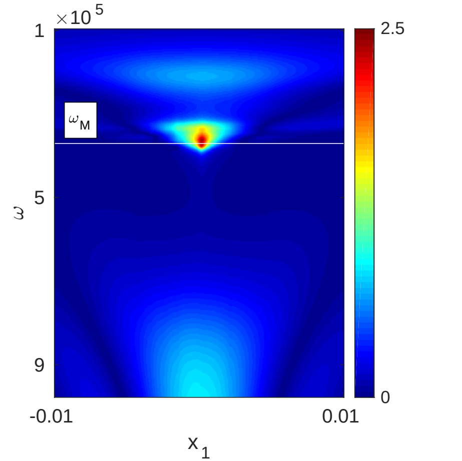
$\Phi = 0$. The white line indicates the Minnaert resonance frequency.

A priori evaluation of focusing performance - $\mathrm{Im}(G_{eff})$



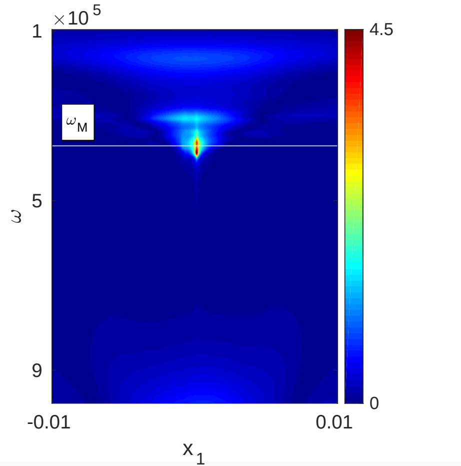
$\Phi = 0.5 \times 10^{-4}$

A priori evaluation of focusing performance - Im(G_{eff})



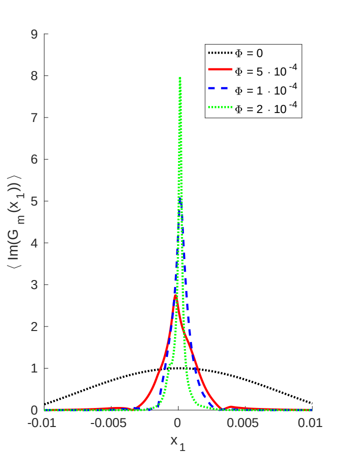
$\Phi = 1 \times 10^{-4}$

A priori evaluation of focusing performance - Im(G_{eff})



$\Phi = 2 \times 10^{-4}$

A priori evaluation of focusing performance - Im(G_{eff})

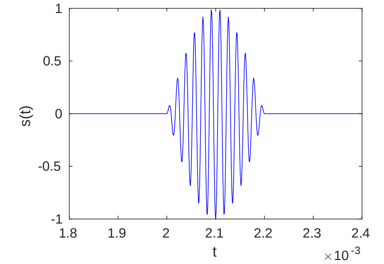


Averaging over frequencies near the Minneart resonance ω_M.

Time-reversal simulation

- **Simulation setup**:
  - We have a point source at the origin x_0 surrounded by a large number of small bubbles.
  - We place four transducers in the far-field at positions x_j for $j = 1, \ldots, 4$, to record and then re-emit signals (time-reversal mirror).
- **Excitation frequency**: Our results for $\mathrm{Im}(G_{eff})$ indicate that super-focusing is achievable at a frequency ω_R just below the Minnaert resonance ω_M.
- **Point source**: We emit a short pulse $s(t)$ from the origin at time $t_0 = 2 \times 10^{-3}$:

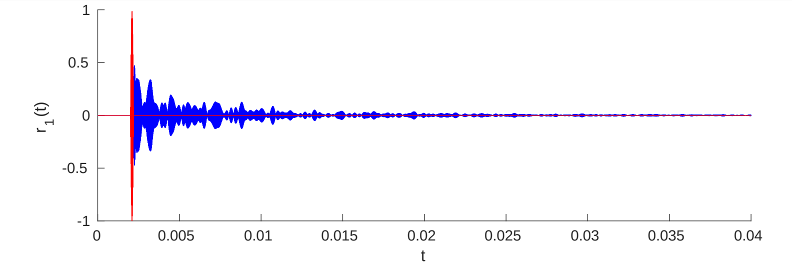
$$s(t) := \sin(\omega_R t) \times \sin\left(\pi \frac{(t - t_0)}{\tau}\right) \times \mathbb{1}\left(0 < t - t_0 < \tau\right).$$



Time-reversal simulation

- **Recording at the time reversal-mirror**: The signal acquired at transducer j is given by:

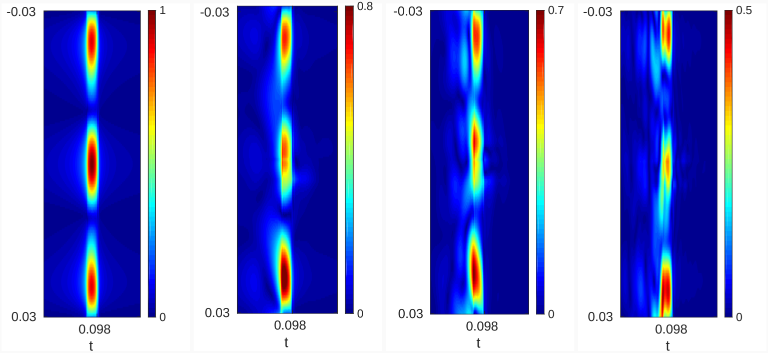
$$r_j(t) = \int \hat{s}(\omega) G_{eff}(x_j - x_0, \omega) e^{-i\omega t} d\omega.$$



- **Time-reversed signal at the origin**: The signal $r_j(-t)$ is emitted from each transducer and the sum of the re-emitted signals gives the time-reversed pressure field.

Time-reversal results (realistic medium)

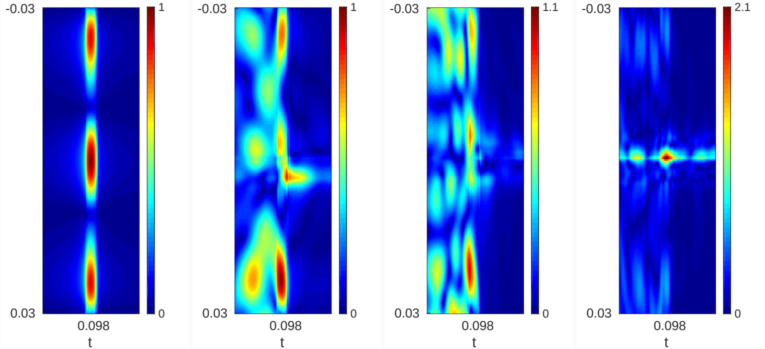
- Due to the length of the recording interval we expect the returning waves to reach the origin at $t = 0.098s$ (horizontal axis). The vertical axis in the plots is the x_1 axis.
- **Time-reversed fields in a realistic medium**: When radiation, thermal, and viscous damping accounted for super-focusing is not achieved.



(a) $\Phi = 0$ **(b)** $\Phi = 0.5 \times 10^{-4}$ **(c)** $\Phi = 1 \times 10^{-4}$ **(d)** $\Phi = 2 \times 10^{-4}$

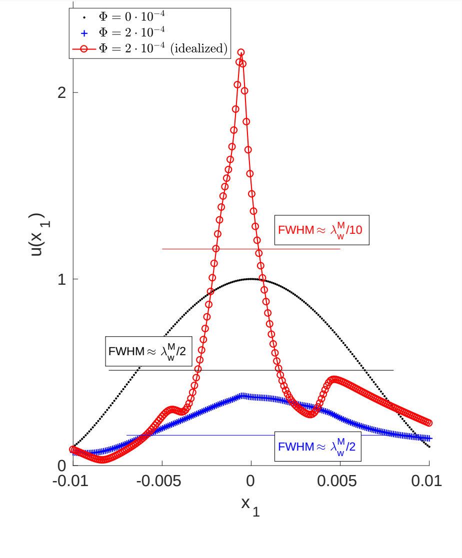
Time-reversal results (idealized medium)

- **Time-reversed fields in an idealistic medium**: When thermal and viscous damping are neglected super-focusing is achieved.



(e) $\Phi = 0$ **(f)** $\Phi = 0.5 \times 10^{-4}$ **(g)** $\Phi = 1 \times 10^{-4}$ **(h)** $\Phi = 2 \times 10^{-4}$

Full-width-at-half-maximum (FWHM) comparison



Super-focusing of acoustic waves with a bubble cloud

- **Without thermal and viscous damping**:
  - The Minnaert resonance ω_M is the optimal excitation frequency when thermal and viscous losses are neglected and represents the focusing enhancement achievable in the idealistic case.
  - A sub-wavelength focal spot of $\sim \lambda/10$ is achieved with a volume fraction of $\Phi = 2 \cdot 10^{-4}$.
- **With thermal and viscous damping**:
  - When thermal and viscous losses are accounted for the scattering enhancement is counteracted by the increased dissipation at the Minnaert resonance and the super-focusing effect is lost - the focusing is no better than can be achieved in homogeneous medium free from bubbles.

A double negative acoustic metamaterial

A double negative acoustic metamaterial

- **Negative effective material parameters**: An acoustic metamaterial in which both the effective bulk modulus and mass density are negative simultaneously, for some given frequency, results in a negative refraction index (NRI).
- **Application**: A NRI allows can be used to build a superlens for which the focusing is not constrained by the diffraction limit.
- **Bubble dimer**: Two closely spaced identical spherical bubbles.
- **Double negative parameters with one resonator type**: In general, two types of resonating objects are required to make each of the material parameters negative. Due to the hybridized Minnaert resonances of a bubble dimer, a negative refractive index is possible using only one type of resonating object - acoustic bubbles.

A double negative acoustic metamaterial

- **Hybridized Minnaert resonances**. One resonance corresponds to a monopole source and the other corresponds to a dipole source.
  - Monopole sources affect the effective mass density.
  - Dipole sources affect the effective bulk modulus.
- **Far-field behaviour of the bubble dimer**: Scattering can be approximated as the sum of a monopole source and a dipole source.
- **Effective bubbly medium**: For a large number of bubble dimers, excitation at a frequency near the hybridized Minnaert resonances leads to a an effective medium with negative effective bulk modulus and effective mass density.
- **Approach**: First we derive the hybridized Minnaert resonances for a single bubble dimer, then we define the effective medium consisting of N dimers.

The bubble dimer problem

$$\begin{cases} \Delta u + k^2 u = 0, \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D}, \\ \Delta u + k_b^2 u = 0, \quad \text{in} \quad D, \\ u|_+ = u|_-, \quad \text{on} \quad \partial D, \\ \delta \dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{\partial u}{\partial \nu}\Big|_-, \quad \text{on} \quad \partial D, \\ u^s := u - u^{in} \text{ satisfies the Sommerfeld R.C. as } |x| \to \infty. \end{cases}$$

- **Wavespeeds**: v and v_b are both of order 1.
- **Two identical spherical bubbles**: $D = D_1 \cup D_2$ where D_1 and D_2 are identical balls of volume 1 and radius r_0, separated by a distance d_0 of order r_0 such that the distance between the centers of the balls is $d_0 + 2r_0$.

Capacitance coeffcients

- Let ψ_1, $\psi_2 \in L^2(\partial D)$ be given by

$$\mathcal{S}_D[\psi_1] = \begin{cases} 1 & \text{on } \partial D_1, \\ 0 & \text{on } \partial D_2, \end{cases} \qquad \mathcal{S}_D[\psi_2] = \begin{cases} 0 & \text{on } \partial D_1, \\ 1 & \text{on } \partial D_2. \end{cases}$$

Then

$$\ker\left(-\frac{1}{2}I + \mathcal{K}_D^*\right) = \text{span}\ \{\psi_1, \psi_2\}.$$

- We define the capacitance coefficients matrix $C = (C_{ij})$ by

$$C_{ij} := -\int_{\partial D_j} \psi_i, \quad i, j = 1, 2.$$

Boundary integral represenation of the problem

- **Boundary integral formulation**: The hybridized Minnaert resonances of the bubble dimer D are the two smallest ω such that there exists a non-trivial solution to the following equation:

$$\mathcal{A}_\omega^\delta[\Psi] = 0,$$

where $\psi_b, \psi \in L^2(\partial D)$ such that

$$\mathcal{A}_\omega^\delta := \mathcal{A}(\omega, \delta) = \begin{pmatrix} \mathcal{S}_D^{k_b} & -\mathcal{S}_D^{k} \\ -\frac{1}{2} + \mathcal{K}_D^{k_b,*} & -\delta(\frac{1}{2} + \mathcal{K}_D^{k,*}) \end{pmatrix}, \Psi = \begin{pmatrix} \psi_b \\ \psi \end{pmatrix}.$$

- As in the single bubble problem, $\omega = 0$ is a characteristic value of the limiting operator $\mathcal{A}_\omega^0$.
- The hybridized Minnaert resonances can be found by asymptotically expanding $\mathcal{A}_\omega^\delta$ and solving $\mathcal{A}_\omega^\delta[\Psi] = 0$ for ω such that Ψ is non-trivial.

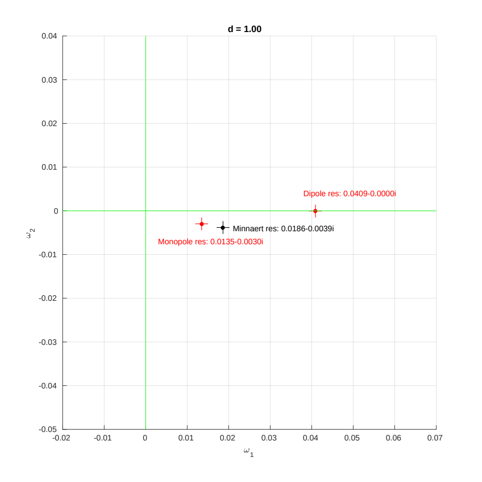
Hybridized Minnaert resonances for the bubble dimer

- **Hybridized Minnaert resonances for the bubble dimer**:

$$\omega_1 = \sqrt{v_b^2(C_{11} + C_{12})}\delta^{1/2} - i\tau_1\delta + O(\delta^{3/2}),$$
$$\omega_2 = \sqrt{v_b^2(C_{11} - C_{12})}\delta^{1/2} + O(\delta^{3/2}).$$

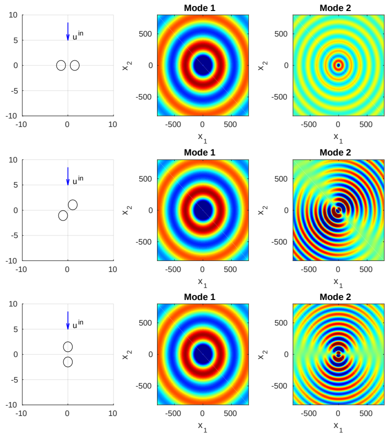
- **Simultaneous excitation**: ω_1 and ω_2 are close together so we can excite both resonances simultaneously.

Hybridized Minnaert resonances for the bubble dimer



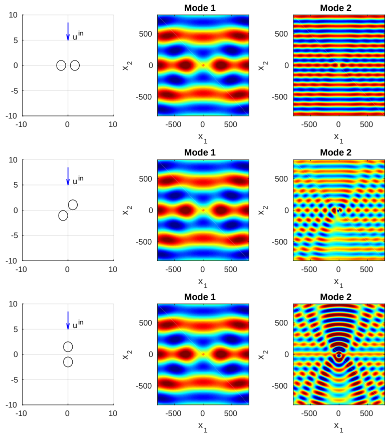
The hybridized Minnaert resonances ω_1 and ω_2 and the single bubble Minnaert resonance ω_M.

Scattered fields for the bubble dimer



Scattered fields for the bubble dimer when it is excited at the monopole mode ω_1 and the dipole mode ω_2.

Total fields for the bubble dimer



Total fields due to the bubble dimer when it is excited at the monopole mode ω_1 and the dipole mode ω_2.

Far-field behaviour of the bubble dimer

- In the far-field the scattered field can be approximated as $\delta \to 0$ by

$$u^s(x) = g^0(\omega)u^{in}(0)G^k(x) + \nabla u^{in}(0) \cdot g^1(\omega)\nabla G^k(x) + O(\delta|x|^{-1}),$$

where

$$g^0(\omega) := \frac{2(C_{11} + C_{12})}{1 - \omega_1^2/\omega^2}(1 + O(\delta^{1/2})),$$

$$g^1(\omega) = (g^1_{ij}(\omega)),\ g^1_{ij}(\omega) := \int_{\partial D} (\mathcal{S}_D^0)^{-1}[y_i]y_j + \frac{\delta \tilde{v}^2}{|D|(\omega_2^2 - \omega^2)} P^2 \delta_{i,1}\delta_{j,1}.$$

- When the excitation frequency ω is near both ω_1 and $\boldsymbol{\omega_2}$, the monopole and dipole modes both have a significant effect on scattering in the far-field.

Effective medium of bubble dimers

$$
\begin{cases}
\Delta u^N + k^2 u^N = 0, \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{D^N}, \\
\Delta u^N + k_b^2 u^N = 0, \quad \text{in} \quad D^N, \\
u^N|_+ = u^N|_-, \quad \text{on} \quad \partial D^N, \\
\delta \left. \frac{\partial u^N}{\partial \nu} \right|_+ = \left. \frac{\partial u^N}{\partial \nu} \right|_-, \quad \text{on} \quad \partial D^N, \\
u^s := u^N - u^{in} \text{ satisfies the Sommerfeld R.C. as } |x| \to \infty.
\end{cases}
$$

- **Large number of bubble dimers**: D^N represents the domain occupied by $N \gg 1$ identical uniformly distributed and oriented bubble dimers:
$$D^N := \cup_{1 \le j \le N} D_j^N, \qquad D_j^N = z_j^N + sR_{d_j^N} D, \qquad 1 \le j \le N.$$
- The center of dimer j is z_j^N.
- The orientation of dimer j is $R_{d_j^N} D$.
- The dimers all have the same size $s \ll 1$.
- We assume $\{z_J^N : 1 \le j \le N\} \subset \Omega$ where Ω is a bounded domain.
- **Gas concentration**: The bubbly medium is very dilute.

Effective medium of bubble dimers

- **Total field outside the bubbles**:

$$u^N = u^{in} + \sum_{1 \le j \le N} \mathcal{S}^k_{D_j^N}[\psi_j^N].$$

- For $1 \le j \le N$ we have
  - **Field incident** on dimer D_j^N:

$$u_j^{i,N} = u^{in} + \sum_{i \neq j} \mathcal{S}^k_{D_i^N}[\psi_i^N].$$

  - **Field scattered** by D_j^N:

$$u_j^{s,N} = \mathcal{S}^k_{D_j^N}[\psi_j^N].$$

- **Field scattered by dimer *j* well away from the bubbles**: After making several assumptions regarding the distribution and orientation of the bubble dimers, $u_j^{s,N}(x)$ for x sufficiently far from any of the bubbles can be approximated as:

$$u_j^{s,N}(x) \approx u_j^{i,N}(z_j^N) g^0(\omega, \delta, D_j^N) G^k(x - z_j^N) + \nabla u_j^{i,N}(z_j^N) \cdot g^1(\omega, \delta, D_j^N) \nabla G^k(x - z_j^N).$$

.

Effective medium of bubble dimers

- **Approximation of total field for N bubble dimers**:

$$u^N(x) \approx u^{in}(x) + \sum_{1 \le j \le N} g^0(\omega, \delta, D_j^N) u_j^{i,N}(z_j^N) G^k(x - z_j^N)$$
$$+ \sum_{1 \le j \le N} \nabla u_j^{i,N}(z_j^N) \cdot g^1(\omega, \delta, D_j^N) \nabla G^k(x - z_j^N).$$

- **Approximation of total field as $N \to \infty$**:

$$u(x) = u^{in}(x) + \int_\Omega \Lambda \tilde{g}^0 u(y) \tilde{V} G^k(x - y) dy$$
$$+ \int_\Omega \Lambda \tilde{g}^1 \nabla u(y) \tilde{B} \nabla G^k(x - y) dy.$$

Effective medium of bubble dimers

- **Equation for effective medium**: Applying the operator $\triangle + k^2$ to both sides of the approximate total field equation gives

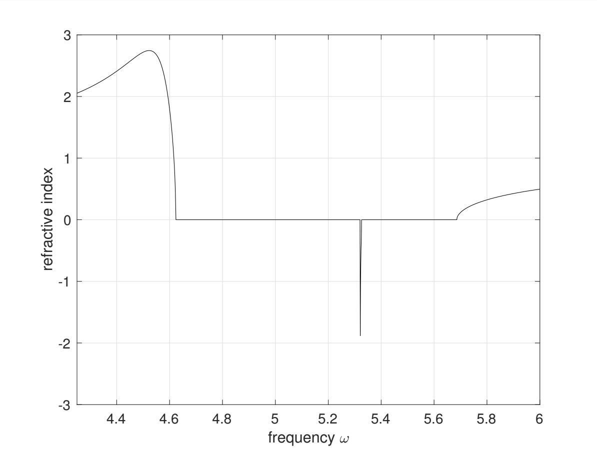
$$(\triangle + k^2)u(x) = \Lambda \tilde{g}^0 \tilde{V} u(x) + \nabla \cdot \left(\Lambda \tilde{B} \nabla u \right)(x), \quad \text{in } \Omega.$$

or equivalently,

$$\nabla \cdot \underbrace{\left(I - \Lambda \tilde{g}^1 \tilde{B} \right)}_{\kappa_{eff}} \nabla u(x) + \underbrace{(k^2 - \Lambda \tilde{g}^0 \tilde{V})}_{\omega^2 \rho_{eff}} u(x) = 0, \quad \text{in } \Omega.$$

- **Effective material parameters**: ρ_{eff} and κ_{eff} can both be negative simultaneously when the bubbly medium is excited near the hybridized Minnaert resonances.

Double negative material parameters



The effective material parameters both become negative when excited near the hybridized Minnaert resonances ω_1 and ω_2 leading to a negative refractive index.

Concluding remarks

- **Key mechanism**: Acoustic bubbles that are much smaller than the incident wavelength are effectively invisible to incoming waves. However, near Minnaert resonance frequencies scattering from acoustic bubbles is greatly enhanced.
- **Applications**: The resonant enhancement strongly affects overall wave propagation and is the basis for effects such as super-absorption, phononic bandgaps, double negative metamaterials, and sub-wavelength focusing.

The End.