

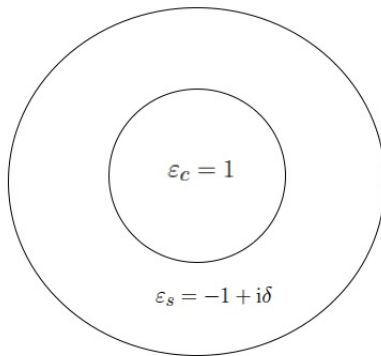
# Plasmonic resonances and cloaking in linear elasticity

Hongyu Liu

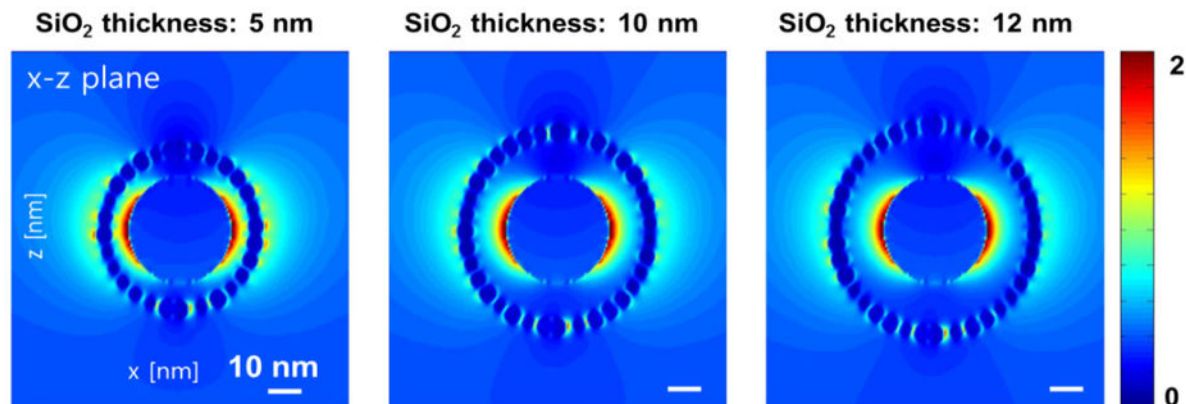
Math Department, Hong Kong Baptist University

Jeju, Korea, Feb 10, 2018

## Anomalous localized resonance and cloaking



- Nicorovici, McPhedran and Milton; Phys. Rev. B., 1994
- Milton and Nicorovici; Proc. Royal Soc., 2006



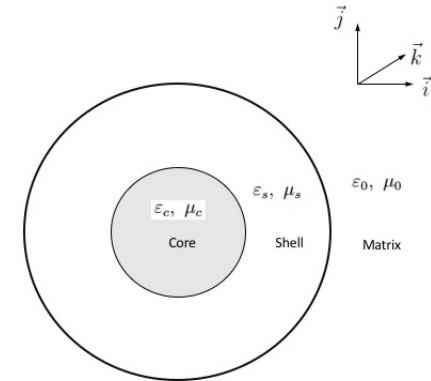
Kochuveedu etc. Scientific Reports, 2013

## Anomalous Localized Resonance

- **Anomalous resonance**: associated to the infinite kernel of a non-elliptic PDO.
- **Localized**: Highly oscillatory+energy blowup+sharp boundaries between different regions
- **Cloaking effect**
- **Surprisingly interesting**: strong dependence on the source location and connection to the spectrum of the Neumann-Poincaré operator!
- Ammari, Ciraolo, Kang, Lee, and Milton; ARMA ,2013.

## Transverse electromagnetic waves and Helmholtz system

$$\left\{ \begin{array}{l} \nabla \cdot (\epsilon(x) \nabla u(x)) + k^2 q(x) u(x) = f(x) \quad x \in \mathbb{R}^d, \\ \lim_{|x| \rightarrow +\infty} |x|^{(N-1)/2} \left( \frac{x}{|x|} \cdot \nabla u - iku \right) = 0. \end{array} \right.$$



Mathematical convenience:

$$i). \quad q \equiv 1$$

*ii*). the loss parameter/regularization parameter is attached to  $\epsilon$

## Core-Shell-Matrix Structure and Electrostatics

- Core-Shell-Matrix structure ( $d = 2, 3$ )

$$\epsilon_0(x) = \begin{cases} \epsilon_c(x), & x \in \Sigma, \\ \epsilon_s(x), & x \in \Omega \setminus \bar{\Sigma}, \\ \epsilon_m(x), & x \in \mathbb{R}^d \setminus \bar{\Omega}. \end{cases}$$

$$\epsilon_\delta(x) = \epsilon_0(x) + i\delta\chi_D(x), \quad x \in \mathbb{R}^d, \quad d = 2, 3$$

- Plasmon constant:  $\epsilon_s < 0$  and  $\delta > 0$  : loss parameter.
- Wave scattering in the quasi-static regime (nano-scale)

Wave pressure  $u(x) = u_\delta(x) \in \mathbb{C}$ :

$$\begin{cases} \nabla_x \cdot (\epsilon_\delta(x)(\nabla_x u_\delta(x))) = f(x), & x \in \mathbb{R}^d, \\ u_\delta(x) = o(1), \quad d = 2; \quad \mathcal{O}(\|x\|^{-1}), \quad d = 3, & \|x\| \rightarrow +\infty, \end{cases}$$

where  $f(x)$  denotes a source term that is compactly supported in  $\mathbb{R}^d \setminus \overline{\Omega}$  and satisfies

$$\int_{\mathbb{R}^d} f(x) dx = 0.$$

- Cloaking due to anomalous localized resonance (CALR)

$$\mathcal{E}_\delta[u] := \delta \int_{\Omega \setminus D} |\nabla u(x)|^2 dx \rightarrow +\infty \quad \text{as } \delta \rightarrow 0,$$

and

$$|u(x)| \leq C \quad \text{for } |x| \geq R',$$

## Non-elliptic PDO with Nontrivial Kernel

- Non-elliptic PDO

$$\mathcal{L}_{\epsilon_0}\psi := \nabla \cdot (\epsilon_0 \nabla \psi)$$

Recall:

$$\epsilon_0(x) = \begin{cases} \epsilon_c(x), & x \in \Sigma, \\ \epsilon_s(x), & x \in \Omega \setminus \bar{\Sigma}, \\ \epsilon_m(x), & x \in \mathbb{R}^d \setminus \bar{\Omega}. \end{cases}$$

with  $\epsilon_s < 0$ .

- $\mathcal{L}_{\epsilon_0}\psi = 0$ ,  $\psi(x) \rightarrow +0$  as  $\|x\| \rightarrow +0$ . Nontrivial  $\psi$ ?
- $\text{Ker}(\mathcal{L}_{\epsilon_0}) \neq \emptyset$ : resonance can be induced

## Spectrum of Neumann-Poincaré Operator

- Consider

$$\epsilon_0(x) = \begin{cases} c, & \|x\| < r_e, \\ 1, & \|x\| > r_e. \end{cases}$$

and  $\mathcal{L}_{\epsilon_0}\psi = 0$ ,  $\psi(x) \rightarrow +0$  as  $\|x\| \rightarrow +\infty$ .

- Integral ansatz

$$\psi(x) = S[\varphi](x) := \int_{\partial B_{r_e}} \Phi(x - y)\varphi(y) ds(y).$$

- Match transmission condition on  $\partial B_{r_e}$ :

$$K^*[\varphi] = \frac{c+1}{2(c-1)}\varphi, \quad K^* := \partial_\nu S[\varphi]$$

- The choice of  $c$  is critical!

## Classical Plasmonic Structure

- Classical structure

$$\epsilon_0(x) = \begin{cases} 1, & x \in B_{r_c}, \\ -1, & x \in B_{r_e} \setminus B_{r_c}, \\ 1, & x \in \mathbb{R}^d \setminus B_{r_e}. \end{cases}$$

- Ammari, Ciraolo, Kang, Lee, and Milton; ARMA, 2013.
- **Two dimensions+radial geometry**: requirement on the exact spectral information of the Neumann-Poincaré operator.
- **Can deal with**: resonance and non-resonance+dependence on source location+localizing and cloaking effects

- Critical radius:  $r^* = \sqrt{r_e^3/r_i}$
- Extension to ellipse geometry.
- The classical structure **fails!** (Ammari, Ciraolo, Kang, Lee and Milton, 2013)

$$\epsilon_0(x) = \begin{cases} 1, & x \in B_{r_c}, \\ -1, & x \in B_{r_e} \setminus B_{r_c}, \\ 1, & x \in \mathbb{R}^d \setminus B_{r_e}. \end{cases}$$

Indeed,  $-1$  can be replaced by any negative constant  $c$  !

- Kohn, Lu, Schweizer and Weinstein; Variational approach, CMP 2014.

○ 3D plasmon resonance in electrostatics: Li, Li and Liu, SIAP, 2015.

● **CRUX**: negative  $c$  varies according to  $\delta$ .

● Typical example:  $c(\delta) = -1 - n(\delta)^{-1}$ .

● CALR is sensitive to metamaterial parameters.

# Plasmon Resonances with Finite Frequencies: Quasi-static Approximation

---

(Ando, Kang and Liu; SIAP, 2016)

- Finite frequency regime

$$\nabla \cdot (\epsilon_\delta \nabla u_\delta) + k^2 u_\delta = f$$

- Two asymptotic processes:

$$\delta \rightarrow +0 \quad \text{and} \quad k \rightarrow +0.$$

# Plasmon Resonances at Optical Frequencies

(Li-Liu, 2017)

- Neumann-Poincaré operator

$$\left(K_{\partial B_R}^k\right)^* [\varphi](x) = \text{p.v.} \int_{\partial B_R} \frac{\partial}{\partial \nu_x} G^k(x-y) \varphi(y) ds(y) \quad x \in \partial B_R,$$

$G_k$ : fundamental solution to  $\Delta + k^2$

## Theorem

$$\left(K_{\partial B_R}^k\right)^* [Y_m](x) = \lambda_{m,k,R} Y_m(\hat{x}), \text{ non self-adjoint}$$

where

$$\lambda_{m,k,R} = \frac{1}{2} - ik^2 R^2 j_n'(kR) h_n^{(1)}(kR) = -\frac{1}{2} - ik^2 R^2 j_n(kR) h_n^{(1)'}(kR),$$

and on the boundary  $\partial B_R$ ,

$$S_{\partial B_R}^k[Y_m](x) = \chi_{m,k,R} Y_m(\hat{x}), \quad x \in \partial B_R,$$

where

$$\chi_{m,k,R} = -ikR^2 h_m^{(1)}(kR) j_m(kR).$$

- Core-shell-structure: a bit complicated, but can induce anomalous localized resonance and cloaking.

$$\left\{ \begin{array}{l} \nabla \cdot (\epsilon(x) \nabla u(x)) + k^2 q(x) u(x) = f(x) \quad x \in \mathbb{R}^N, \\ \lim_{|x| \rightarrow +\infty} |x|^{(N-1)/2} \left( \frac{x}{|x|} \cdot \nabla u - iku \right) = 0. \end{array} \right.$$

with core-shell-matrix material structure

$$(\epsilon, q) = (\epsilon_c, 1)\chi(B_{r_i}) + (\epsilon_s + i\delta, 1)\chi(B_{r_e} \setminus \overline{B_{r_i}}) + (1, 1)\chi(\mathbb{R}^N \setminus \overline{B_{r_e}}).$$

○ Newtonian potential of the source

$$F(x) = \int_{\mathbb{R}^3} G^k(x - y)f(y)dy = \sum_{n=N}^{\infty} \beta_n j_n(kr)Y_n(\hat{x}),$$

**Theorem** Suppose the source  $f(x)$  supported inside the critical radius  $r_* = \sqrt{k^2 r_e^3 / r_i}$  and its Newtonian potential  $F$  satisfies the constraint. Then for any  $M \in \mathbb{R}_+$ , there exists  $n_0 \in \mathbb{N}_+$  such that if the plasmon configuration is chosen as follows

$$\epsilon_s = -1 - 1/n_0, \quad \epsilon_c = (1 + 1/n_0)^2, \quad \text{and} \quad \delta = \rho^{n_0},$$

with  $\rho = r_i/r_e$ , then  $\mathcal{E}_\delta \geq M$ . Moreover, one has that as  $M \rightarrow \infty$ ,  $n_0 \rightarrow \infty$  (or equivalently,  $\delta \rightarrow +0$ ). That is, the configuration is resonant as  $\delta \rightarrow +0$ . Moreover, when  $|x| \geq r_e^2/r_i$ , the solution associated with the configuration described above is bounded. That is, CALR occurs. If the source  $f(x)$  is supported outside the critical radius  $r_* = \sqrt{k^2 r_e^3 / r_i}$ , then plasmon resonance does not occur.

- Similar results for 2D.
- Can hold for more general scenarios of  $\delta \rightarrow \delta_0$  with  $\delta_0$  not being zero.
- Metamaterial parameters can also be given in terms of Drude or Lorentz models.

## Plasmon Resonances in Elastostatics: Elastic Material Tensor

- Elastic material tensor

$\mathbf{C}_{\lambda,\mu}(x) := (C_{ijkl}(x))_{i,j,k,l=1}^d$ ,  $x \in \mathbb{R}^d$  with  $d = 2, 3$ , be a four-rank tensor such that

$$C_{ijkl}(x) := \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad x \in \mathbb{R}^N,$$

where  $\lambda, \mu \in \mathbb{R}$ .

- Strong convexity condition

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0.$$

## Plasmon Resonances in Elastostatics: Lamé System

- **Plasmonic structure** (Ando, Ji, Kang, Kim and Yu; EJAM, 2015)

$$\left(\tilde{\lambda}(x), \tilde{\mu}(x)\right) = \left(A(x) + i\delta\right)(\lambda, \mu), \quad x \in \mathbb{R}^d,$$

$$A(x) = \begin{cases} +1, & x \in \Sigma, \\ \mathbf{c}, & x \in \Omega \setminus \bar{\Sigma}, \\ +1, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

- **Lamé system for  $\mathbf{u}_\delta(x) \in \mathbb{C}^d \cap H_{loc}^1(\mathbb{R}^d)^d$**

$$\left\{ \begin{array}{l} \mathcal{L}_{\tilde{\lambda}, \tilde{\mu}} \mathbf{u}_\delta(x) = \mathbf{f}(x), \quad x \in \mathbb{R}^d, \\ \mathbf{u}_\delta|_- = \mathbf{u}_\delta|_+, \quad \partial_{\nu_{\tilde{\lambda}, \tilde{\mu}}} \mathbf{u}_\delta|_- = \partial_{\nu_{\tilde{\lambda}, \tilde{\mu}}} \mathbf{u}_\delta|_+ \quad \text{on } \partial\Sigma \cup \partial\Omega, \\ \mathbf{u}_\delta(x) = \mathcal{O}\left(\|x\|^{-1}\right) \quad \text{as } \|x\| \rightarrow +\infty, \end{array} \right.$$

where  $\mathbf{f} = (f_l)_{l=1}^N \in \mathbb{R}^d \cap H^{-1}$  with  $\langle f_l, \mathbf{1} \rangle = 0$ ; and

$$\mathcal{L}_{\tilde{\lambda}, \tilde{\mu}} \mathbf{u}_\delta := \nabla \cdot \left( \mathbf{C}_{\tilde{\lambda}, \tilde{\mu}} \nabla^s \mathbf{u}_\delta \right) = \tilde{\mu} \Delta \mathbf{u}_\delta + (\tilde{\lambda} + \tilde{\mu}) \nabla \nabla \cdot \mathbf{u}_\delta$$

and  $\nabla^s \mathbf{u}_\delta := \frac{1}{2} (\nabla \mathbf{u}_\delta + \nabla \mathbf{u}_\delta^T)$ . The conormal derivative (or traction) is defined by

$$\partial_{\boldsymbol{\nu}_{\tilde{\lambda}, \tilde{\mu}}} \mathbf{u}_\delta = \frac{\partial \mathbf{u}_\delta}{\partial \boldsymbol{\nu}_{\tilde{\lambda}, \tilde{\mu}}} := \tilde{\lambda} (\nabla \cdot \mathbf{u}_\delta) \boldsymbol{\nu} + \tilde{\mu} (\nabla \mathbf{u}_\delta + \nabla \mathbf{u}_\delta^T) \boldsymbol{\nu} \quad \text{on } \partial \Sigma \text{ or } \partial \Omega,$$

## Plasmon Resonances in Elastostatics

- Bilinear Form

$$\mathbf{P}_{\lambda,\mu}(\mathbf{u}, \mathbf{v}) := \int_{\mathbb{R}^d} \left[ \lambda(\nabla \cdot \mathbf{u})(\overline{\nabla \cdot \mathbf{v}})(x) + 2\mu \nabla^s \mathbf{u} : \overline{\nabla^s \mathbf{v}}(x) \right] dV(x),$$

where for two matrices  $\mathbf{A} = (a_{ij})_{i,j=1}^d$  and  $\mathbf{B} = (b_{ij})_{i,j=1}^d$ ,

- Elastic Energy

$$\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^d a_{ij} b_{ij}.$$

$$\mathbf{E}_\delta(\mathbf{C}_{\tilde{\lambda}, \tilde{\mu}}, \mathbf{f}) := \frac{\delta}{2} \mathbf{P}_{\lambda,\mu}(\mathbf{u}_\delta, \mathbf{u}_\delta),$$

which is the imaginary part of

$$\frac{1}{2} \int_{\mathbb{R}^N} \overline{\nabla^s \mathbf{u}_\delta} : \mathbf{C}_{\tilde{\lambda}, \tilde{\mu}} \nabla^s \mathbf{u}_\delta dV.$$

$\mathbf{E}_\delta$  signifies the energy dissipation of the elastostatic system.

## Plasmon Resonances in Elastostatics

The configuration  $(\mathbf{C}_{\tilde{\lambda}, \tilde{\mu}}, \mathbf{f})$  is said to be resonant if

$$\limsup_{\delta \rightarrow +0} \mathbf{E}_\delta(\mathbf{C}_{\tilde{\lambda}, \tilde{\mu}}, \mathbf{f}) = +\infty.$$

Recall:

$$(\tilde{\lambda}(x), \tilde{\mu}(x)) = (A(x) + i\delta)(\lambda, \mu), \quad x \in \mathbb{R}^d,$$

$$A(x) = \begin{cases} +1, & x \in \Sigma, \\ c, & x \in \Omega \setminus \bar{\Sigma}, \\ +1, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

## Plasmon resonances in elastostatics: our results

- Choice of plasmon parameter  $c$  to induce resonances.
- Dependence on the location and form of the source;
- Variational argument;
- Spectral properties of the Neumann-Poincaré operator in elastostatics. (**Non compact**)

## Plasmon Resonances in Elastostatics: Variational Principles

- Decomposition

$$\mathbf{u}_\delta = \mathbf{v}_\delta + i\frac{1}{\delta}\mathbf{w}_\delta,$$

- Decoupled PDE systems

$$\begin{aligned}\mathcal{L}_{\lambda_A, \mu_A} \mathbf{v}_\delta - \mathcal{L}_{\lambda, \mu} \mathbf{w}_\delta &= \mathbf{f}, \\ \mathcal{L}_{\lambda_A, \mu_A} \mathbf{w}_\delta + \delta^2 \mathcal{L}_{\lambda, \mu} \mathbf{v}_\delta &= \mathbf{0},\end{aligned}$$

where

$$(\lambda_A(x), \mu_A(x)) := A(x)(\lambda, \mu), \quad x \in \mathbb{R}^d$$

- Energy functionals

$$\mathbf{I}_\delta(\mathbf{v}, \mathbf{w}) := \frac{\delta}{2} \mathbf{P}_{\lambda, \mu}(\mathbf{v}, \mathbf{v}) + \frac{1}{2\delta} \mathbf{P}_{\lambda, \mu}(\mathbf{w}, \mathbf{w}) \quad \text{for } (\mathbf{v}, \mathbf{w}) \in \mathcal{S} \times \mathcal{S},$$

$$\mathbf{J}_\delta(\mathbf{v}, \boldsymbol{\psi}) := \int_{\mathbb{R}^3} \mathbf{f} \cdot \boldsymbol{\psi} - \frac{\delta}{2} \mathbf{P}_{\lambda, \mu}(\mathbf{v}, \mathbf{v}) - \frac{\delta}{2} \mathbf{P}_{\lambda, \mu}(\boldsymbol{\psi}, \boldsymbol{\psi}) \quad \text{for } (\mathbf{v}, \boldsymbol{\psi}) \in \mathcal{S} \times \mathcal{S}.$$

where the Banach space

$$\mathcal{S} := \left\{ \mathbf{u} \in H_{\text{loc}}^1(\mathbb{R}^3)^3; \nabla \mathbf{u} \in L^2(\mathbb{R}^3)^{3 \times 3} \text{ and } \int_{B_{R_0}} \mathbf{u} = 0 \right\},$$

endowed with the Sobolev norm for  $\mathbf{u} = (u_i)_{i=1}^d$ ,

$$\|\mathbf{u}\|_{\mathcal{S}} := \left( \int_{\mathbb{R}^d} \sum_{i=1}^d \|\nabla u_i\|^2 dV + \int_{B_{R_0}} \|\mathbf{u}\|^2 dV \right)^{1/2}.$$

- Primal variational principle

Minimize  $\mathbf{I}_\delta(\mathbf{v}, \mathbf{w})$  over all pairs  $(\mathbf{v}, \mathbf{w}) \in \mathcal{S} \times \mathcal{S}$   
subject to the PDE constraint  $\mathcal{L}_{\lambda_A, \mu_A} \mathbf{v} - \mathcal{L}_{\lambda, \mu} \mathbf{w} = \mathbf{f}$ ;

The infimum

$$\inf \left\{ \mathbf{I}_\delta(\mathbf{v}, \mathbf{w}); \mathcal{L}_{\lambda_A, \mu_A} \mathbf{v} - \mathcal{L}_{\lambda, \mu} \mathbf{w} = \mathbf{f} \right\}$$

is attainable at a pair  $(\mathbf{v}_\delta, \mathbf{w}_\delta) \in \mathcal{S} \times \mathcal{S}$ . The minimizing pair  $(\mathbf{v}_\delta, \mathbf{w}_\delta)$  verifies that the function  $\mathbf{u}_\delta := \mathbf{v}_\delta + i\delta^{-1}\mathbf{w}_\delta$  is the unique solution to the elastic problem and moreover one has

$$\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{I}_\delta(\mathbf{v}_\delta, \mathbf{w}_\delta) \leq \mathbf{I}_\delta(\mathbf{v}, \mathbf{w})$$

Can be used to show the non-resonance result!

- Dual variational principle

Maximize  $\mathbf{J}_\delta(\mathbf{v}, \psi)$  over all pairs  $(\mathbf{v}, \psi) \in \mathcal{S} \times \mathcal{S}$   
subject to the PDE constraint  $\mathcal{L}_{\lambda_A, \mu_A} \psi + \delta \mathcal{L}_{\lambda, \mu} \mathbf{v} = \mathbf{0}$ .

The supremum

$$\sup \left\{ \mathbf{J}_\delta(\mathbf{v}, \psi); \mathcal{L}_{\lambda_A, \mu_A} \psi + \delta \mathcal{L}_{\lambda, \mu} \mathbf{v} = \mathbf{0} \right\}$$

is attainable at a pair  $(\mathbf{v}_\delta, \psi_\delta) \in \mathcal{S} \times \mathcal{S}$ . The maximizing pair  $(\mathbf{v}_\delta, \psi_\delta)$  verifies that the function  $\mathbf{u}_\delta := \mathbf{v}_\delta + i\psi_\delta$  is the unique solution to the elastic problem, and moreover one has

$$\mathbf{E}_\delta(\mathbf{u}_\delta) = \mathbf{J}_\delta(\mathbf{v}_\delta, \psi_\delta) \geq \mathbf{J}_\delta(\mathbf{v}, \psi).$$

Can be used to show the resonance result!

## Trial functions: perfect plasmon waves

- Non-elliptic PDO and its kernel

$$\left\{ \begin{array}{l} \mathcal{L}_{\lambda_A, \mu_A} \psi = 0, \\ \psi|_- = \psi|_+, \quad \partial_{\nu_{\lambda_A, \mu_A}} \psi|_- = \partial_{\nu_{\lambda_A, \mu_A}} \psi|_+ \quad \text{on } \partial B_R, \\ \psi(x) = \mathcal{O}(\|x\|^{-1}) \quad \text{as } \|x\| \rightarrow \infty, \end{array} \right.$$

where

$$A(x) = \begin{cases} c, & \|x\| \leq R, \\ +1, & \|x\| > R. \end{cases}$$

## Trial functions: perfect plasmon waves

- Plasmon constants and infinite dimensional kernel (Li-Liu, SIMA, 2016)

Consider the above PDE system for a function  $\psi \in H_{loc}^1(\mathbb{R}^3)^3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ . Let  $n \in \mathbb{N}$  and  $n \geq 2$  be fixed and set

$$\begin{aligned}\zeta_1 &:= -1 - \frac{3}{n-1}, \\ \zeta_2 &:= -\frac{(2n+2)((n-1)\lambda + (3n-2)\mu)}{(2n^2+1)\lambda + (2+2n(n-1))\mu}, \\ \zeta_3 &:= -\frac{(2n^2+4n+3)\lambda + (2n^2+6n+6)\mu}{2n((n+2)\lambda + (3n+5)\mu)}.\end{aligned}$$

If

$$c = \zeta_1,$$

there exists a non-trivial solution  $\psi = \hat{\psi}_{n,k} \in H_{loc}^1(\mathbb{R}^3)^3$  as follows:

$$\hat{\psi}_{n,k}(x) = \begin{cases} \mathbf{G}^{n,\zeta_1,k} r^n \mathbf{Y}_n(\hat{x}), & r \leq R, \\ \mathbf{G}^{n,\zeta_1,k} \frac{R^{2n+1}}{r^{n+1}} \mathbf{Y}_n(\hat{x}), & r > R, \end{cases}$$

with  $\mathbf{G}^{n,\zeta_1,k}$ ,  $k = 1, 2, \dots, 2n + 1$ , satisfying

$$\mathbf{t}_{n+1}^1 = \mathbf{0} \quad \text{and} \quad \mathbf{t}_{n-1}^3 = \mathbf{0}.$$

If

$$c = \zeta_2,$$

there exists a non-trivial solution  $\psi = \hat{\psi}_{n,k} \in H_{loc}^1(\mathbb{R}^3)^3$  as fol-

lows:

$$\widehat{\psi}_{n,k}(x) = \begin{cases} \mathbf{G}^{n,\zeta_2,k} r^n \mathbf{Y}_n(\hat{x}) - M_n(r^2 - R^2) \begin{bmatrix} \mathbf{t}_{n-1}^3 \mathbf{D}_{n-1}^{(n-2)x_1} \\ \mathbf{t}_{n-1}^3 \mathbf{D}_{n-1}^{(n-2)x_2} \\ \mathbf{t}_{n-1}^3 \mathbf{D}_{n-1}^{(n-2)x_3} \end{bmatrix} r^{n-2} \mathbf{Y}_{n-2}(\hat{x}), \\ \mathbf{G}^{n,\zeta_2,k} \frac{R^{2n+1}}{r^{n+1}} \mathbf{Y}_n(\hat{x}), \end{cases}$$

with  $\mathbf{G}^{n,\zeta_2,k}$ ,  $k = 1, 2, \dots, 2n - 1$ , satisfying

$$\mathbf{t}_{n+1}^1 = \mathbf{0} \quad \text{and} \quad \mathbf{t}_{n-1}^3 \neq \mathbf{0}.$$

If

$$c = \zeta_3,$$

there exists a non-trivial solution  $\psi = \widehat{\psi}_{n,k} \in H_{loc}^1(\mathbb{R}^3)^3$  as fol-

lows:

$$\hat{\psi}_{n,k}(x) = \begin{cases} \mathbf{G}^{n,\zeta_3,k} r^n \mathbf{Y}_n(\hat{x}), \\ \mathbf{G}^{n,\zeta_3,k} \frac{R^{2n+1}}{r^{n+1}} \mathbf{Y}_n(\hat{x}) + k_n (r^2 - R^2) \begin{bmatrix} \mathbf{t}_{n+1}^1 \mathbf{D}_{n+1}^{(n+2)x_1} \\ \mathbf{t}_{n+1}^1 \mathbf{D}_{n+1}^{(n+2)x_2} \\ \mathbf{t}_{n+1}^1 \mathbf{D}_{n+1}^{(n+2)x_3} \end{bmatrix} \frac{R^{2n+1}}{r^{n+3}} \mathbf{Y}_{n+2}(\hat{x}) \end{cases}$$

with  $\mathbf{G}^{n,\zeta_3,k}$ ,  $k = 1, 2, \dots, 2n + 3$ , satisfying

$$\mathbf{t}_{n+1}^1 \neq \mathbf{0} \quad \text{and} \quad \mathbf{t}_{n-1}^3 = \mathbf{0}.$$

- How to get them?

Undetermined coefficients, highly tedious!

## Various resonance and non-resonance results

The force term  $\mathbf{f}(x)$  is a real-valued distributional functional

$$\mathbf{f} = \mathbf{F}\mathcal{H}^2|_{\partial B_q}, \quad \mathbf{F} : \partial B_q \rightarrow \mathbb{R}^3, \quad \mathbf{F} \in L^2(\partial B_q)^3, \quad q \in \mathbb{R}_+,$$

with zero average. The Fourier series expression of  $\mathbf{f}$  is

$$\mathbf{f} = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{2n+1} \gamma_{n,\zeta_1,k} \mathbf{G}^{n,\zeta_1,k} + \sum_{k=1}^{2n-1} \gamma_{n,\zeta_2,k} \mathbf{G}^{n,\zeta_2,k} + \sum_{k=1}^{2n+3} \gamma_{n,\zeta_3,k} \mathbf{G}^{n,\zeta_3,k} \right) \mathbf{f}_n^q,$$

where

$$\begin{aligned} \mathbf{f}_n^q &= \mathbf{Y}_n \mathcal{H}^2|_{\partial B_q}, \\ \gamma_{n,\zeta_i,k} &= \int_{\partial B_1} \mathbf{f}(q\hat{x}) \cdot \overline{\mathbf{G}^{n,\zeta_i,k} \mathbf{Y}_n(\hat{x})} ds(\hat{x}), \end{aligned}$$

## Various resonance and non-resonance results

- Shell-Matrix Structure without a core
  1. Resonance can occur if  $c$  is properly chosen depending on  $f$ .
- Core-Shell-Matrix Structure with an arbitrary core
  1. No resonance for any fixed  $c$ !
  2. Resonance occurs if  $c$  is properly chosen depending on  $\delta$ .
  3. There is a critical radius  $r^* = \sqrt{R^3}$ .

## Spectral properties of the Neumann-Poincaré Operator

- NP operator for elastostatics:  $D = B_{r_0}$

$$\mathbf{K}_D^*[\boldsymbol{\varphi}](\mathbf{x}) = \text{p.v.} \int_{\partial D} \frac{\partial}{\partial \nu_{\mathbf{x}}} \boldsymbol{\Gamma}^\omega(\mathbf{x} - \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}) ds(\mathbf{y}),$$

where  $\boldsymbol{\Gamma} = (\Gamma_{j,k})_{j,k=1}^3$  is the Kelvin matrix of fundamental solutions to the Lamé operator  $\mathcal{L}_{\lambda,\mu}$ :

$$\Gamma_{j,k}(\mathbf{x}) = -\frac{\gamma_1 \delta_{jk}}{4\pi |\mathbf{x}|} - \frac{\gamma_2 \mathbf{x}_j \mathbf{x}_k}{4\pi |\mathbf{x}|^3},$$

with

$$\gamma_1 = \frac{1}{2} \left( \frac{1}{\mu} + \frac{1}{2\mu + \lambda} \right) \quad \text{and} \quad \gamma_2 = \frac{1}{2} \left( \frac{1}{\mu} - \frac{1}{2\mu + \lambda} \right).$$

**Non Compact!**

## Spectral system of the Neumann-Poincaré Operator in $\mathbb{R}^3$ (Deng-Li-Liu, 2017)

**Theorem.** The the eigenvalues of the operator  $\mathbf{K}_D^*$  with  $D = B_{r_0}$  are given by,  $n \geq 1$ ,

$$\begin{aligned}\xi_1^n &= \frac{3}{4n+2}, \\ \xi_2^n &= \frac{-3\lambda + 2\mu(2n^2 + 2n - 3)}{2(\lambda + 2\mu)(4n^2 - 1)}, \\ \xi_3^n &= \frac{3\lambda - 2\mu(2n^2 + 2n - 3)}{2(\lambda + 2\mu)(4n^2 + 8n + 3)},\end{aligned}$$

and the corresponding eigenfunctions are respectively given by

$$\begin{aligned}h_1^{n,m}(\hat{\mathbf{x}}) &= \nabla_{\partial D} Y_n^m(\hat{\mathbf{x}}) \times \boldsymbol{\nu}, \\ h_2^{n,m}(\hat{\mathbf{x}}) &= -\nabla_{\partial D} Y_{n-1}^m(\hat{\mathbf{x}}) + nY_{n-1}^m(\hat{\mathbf{x}})\boldsymbol{\nu}, \\ h_3^{n,m}(\hat{\mathbf{x}}) &= \nabla_{\partial D} Y_{n+1}^m(\hat{\mathbf{x}}) + (n+1)Y_{n+1}^m(\hat{\mathbf{x}})\boldsymbol{\nu},\end{aligned}$$

where  $Y_n^m(\hat{\mathbf{x}})$  with  $-n \leq m \leq n$  are spherical harmonics and the operator  $\nabla_{\partial D}$  denotes the surface gradient.

- Can be used to derive more accurate characterisations of the plasmon resonances in  $\mathbb{R}^3$ , particularly the localizing and cloaking effects.
- Ando, Kang and Miyanishi; polynomially compact, IMRN, 2017

## Novel Elastic Plasmonic Structures

(Li-Li-Liu, 2017)

Recall:

- $(\tilde{\lambda}(x), \tilde{\mu}(x)) = (A(x) + i\delta)(\lambda, \mu), \quad x \in \mathbb{R}^d,$   
$$A(x) = \begin{cases} +1, & x \in \Sigma, \\ c, & x \in \Omega \setminus \bar{\Sigma}, \\ +1, & x \in \mathbb{R}^d \setminus \bar{\Omega}, \end{cases}$$

- Strong convexity conditions

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0.$$

- Clearly, both strong convexity conditions are violated.

## Novel Elastic Plasmonic Structures

(Li-Li-Liu; JMPA, 2017)

- **Question**: what's the appropriate definition of **negativity** in elasticity.
- **Natural to conjecture**: either one of the two strong convexity conditions is broken, there exist certain plasmon structures that can induce resonances.
- This is **positively affirmed**! Furthermore, rigorously verified the **quasi-static approximation** and **localizing & cloaking effects**.

- Ongoing things:

1. Anomalous localized resonance and cloaking for elasticity without the quasi-static approximation.

2.  $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s$ , shear and compressional parts.

Anomalous localized resonance and cloaking associated with only one type of wave.

**THANK YOU**