

Optimal design for shielding or field enhancement in electrostatics and linear elasticity

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Two Problems:

- (1) Concentrating a field into a region.
- (2) Shielding a region from fields.



Sharp corners concentrate fields

Large Fields also very important for Raman Spectroscopy:

Effect goes as the 4th power of the field intensity.

Well known that rough surfaces enhance Raman Spectroscopy, by orders of magnitude (SERS)

Shielding: Think of Faraday cage to shield Electromagnetic Field,
Shielding from Magnetic Fields, Thermal Currents
Shielding from Vibrations, Sonar

How to measure this?

Threshold exponents on L^γ integrability:

$$\gamma^- \equiv \inf_{\gamma} : \frac{1}{V(\Omega)} \int_{\Omega} |\mathbf{E}(\mathbf{x})|^\gamma d\mathbf{x} < \infty$$

$$\gamma^+ \equiv \sup_{\gamma} : \frac{1}{V(\Omega)} \int_{\Omega} |\mathbf{E}(\mathbf{x})|^\gamma d\mathbf{x} < \infty$$

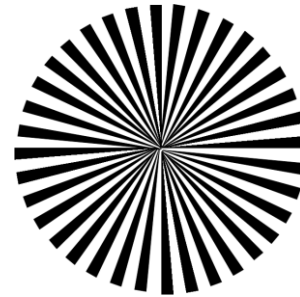
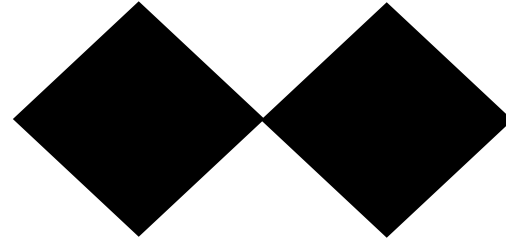
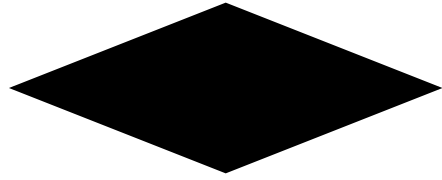
Equivalently, given a (possibly disconnected) subregion $Q \subset \Omega$ of small subvolume $|Q|$ one can maximize or minimize

$$\int_Q |\mathbf{E}(\mathbf{x})|^2 d\mathbf{x}$$

and ask how this depends on $|Q|$ asymptotically as $|Q| \rightarrow 0$

Two isotropic conductors, conductivities σ_1, σ_2 .
Uniform field at infinity

Some Candidates:

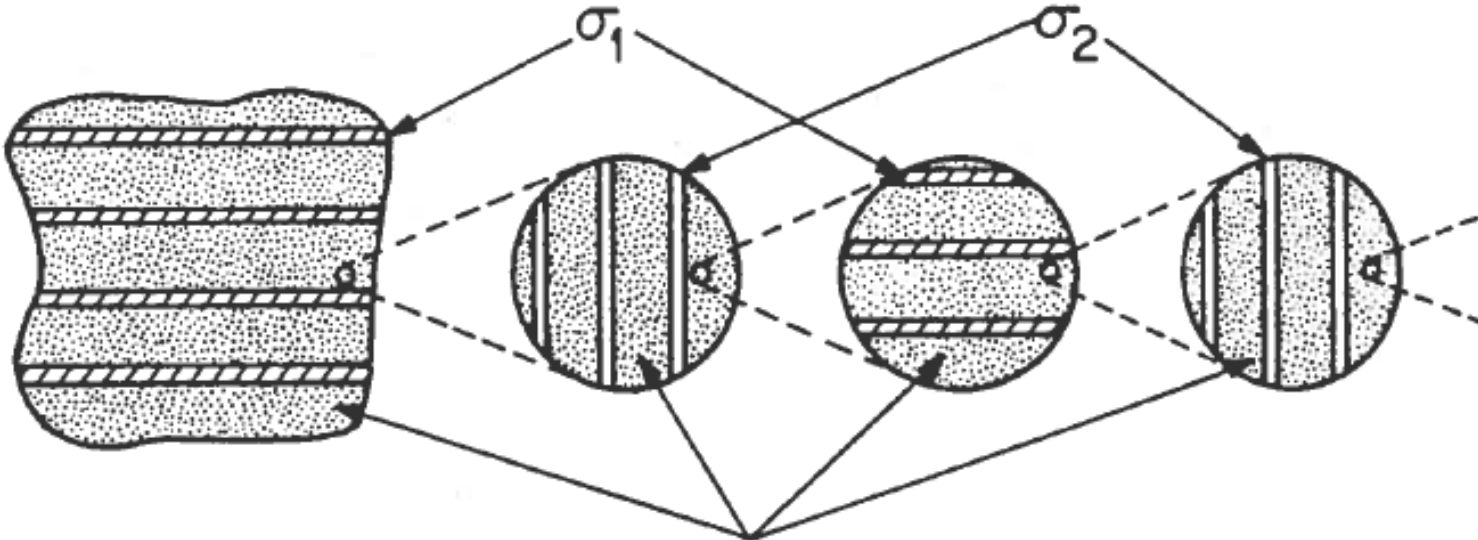


Allow for multiscale inclusions:

$$\gamma^- \equiv \inf_{\gamma} : \frac{1}{V(\Omega)} \int_{\Omega} d\mathbf{x} \int_{Y^n} d\mathbf{y}_1, \dots, d\mathbf{y}_n |\mathbf{E}(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n)|^{\gamma} < \infty$$

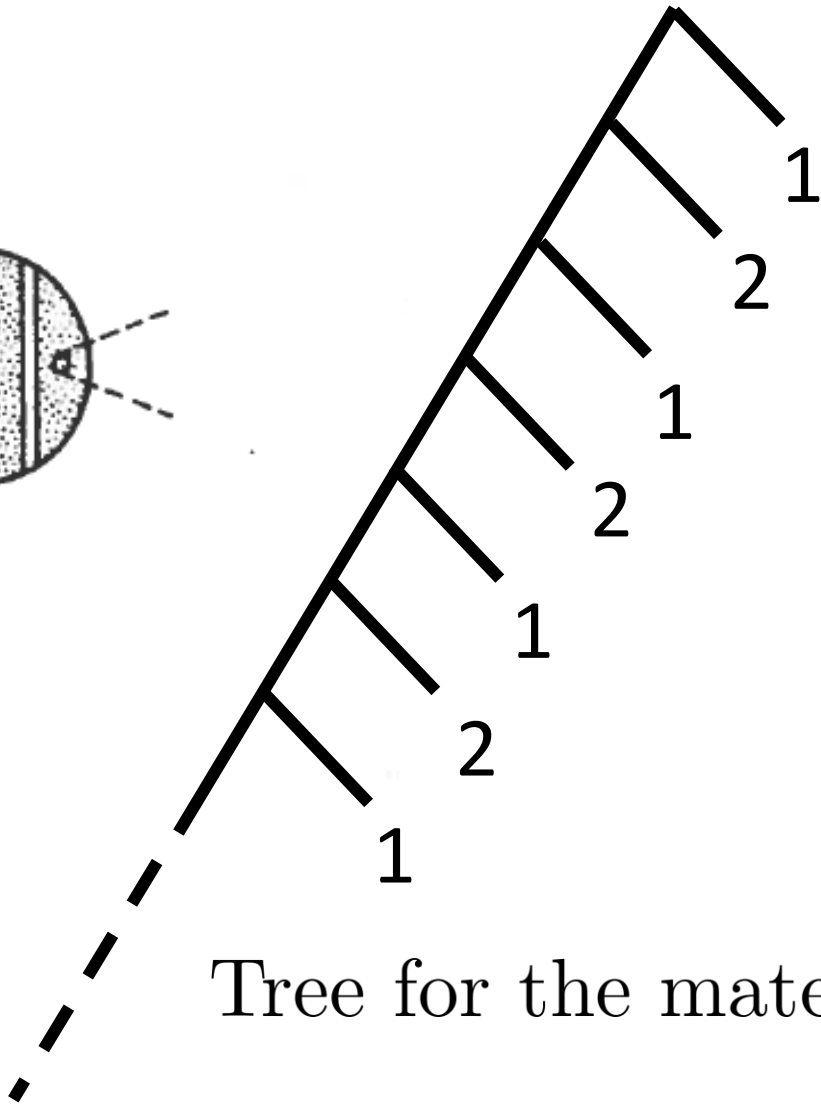
$$\gamma^+ \equiv \sup_{\gamma} : \frac{1}{V(\Omega)} \int_{\Omega} d\mathbf{x} \int_{Y^n} d\mathbf{y}_1, \dots, d\mathbf{y}_n |\mathbf{E}(\mathbf{x}, \mathbf{y}_1, \dots, \mathbf{y}_n)|^{\gamma} < \infty$$

Best:



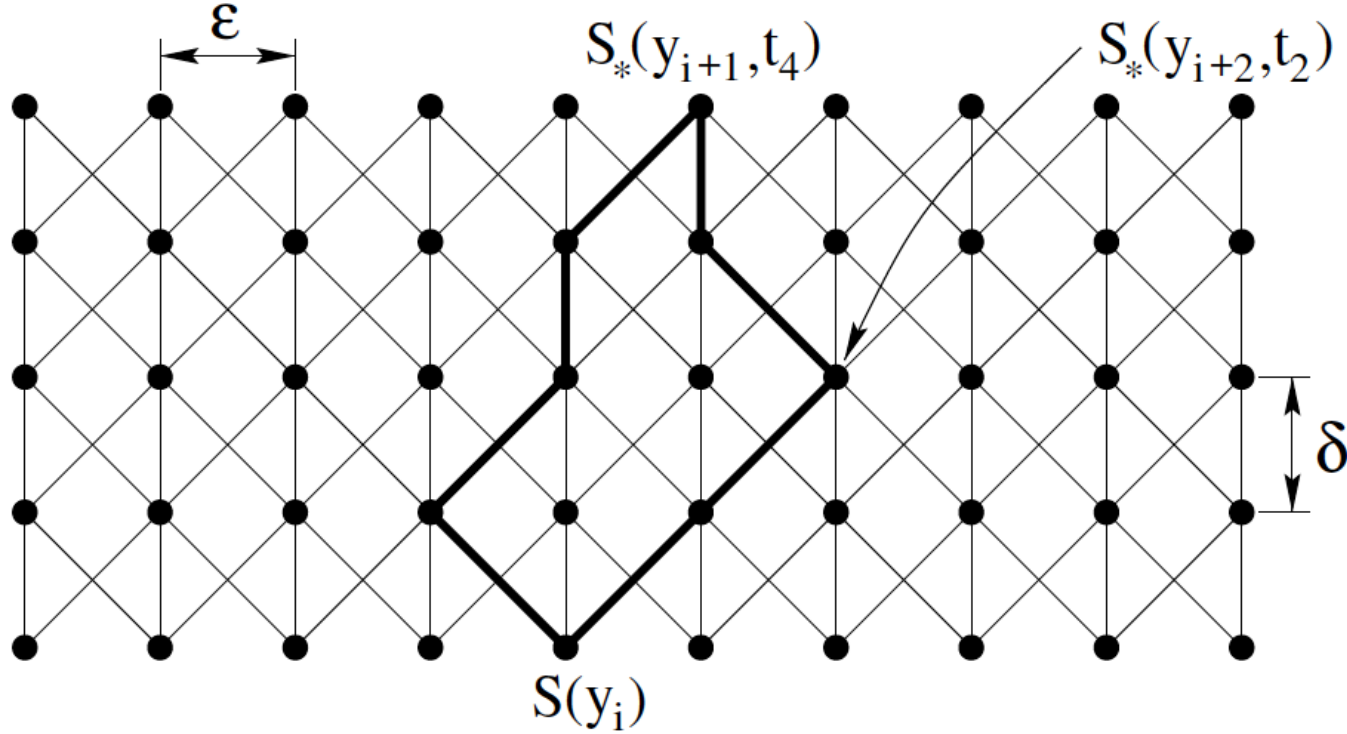
Effective Medium

GWM (1986)



Tree for the material

Totally crazy microstructures: partial differential materials



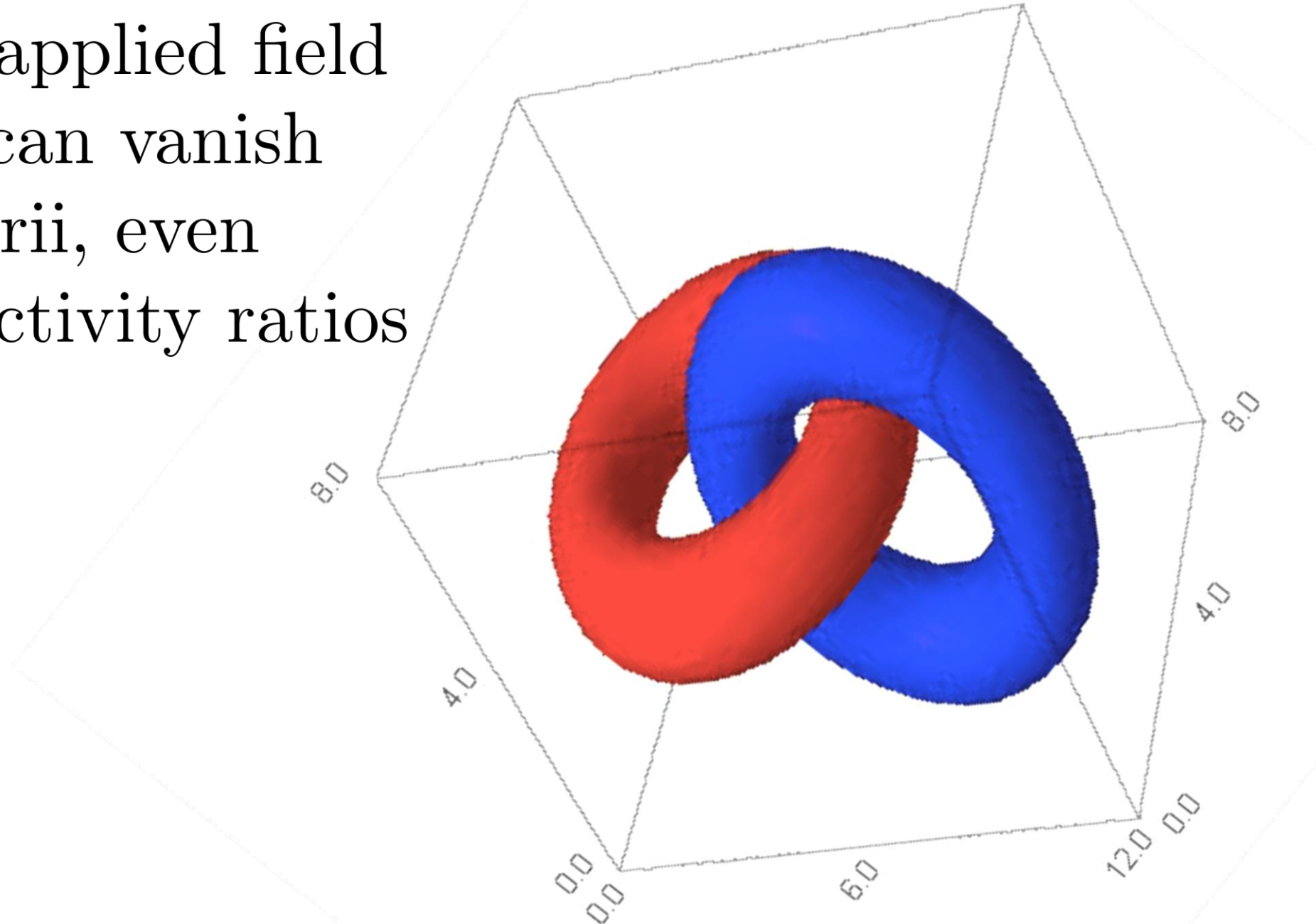
Replace tree by lattice of materials

$$\frac{\partial S_*}{\partial t} = q \left\{ S_* - \Gamma_1(n) - [S_* - \Gamma_1(n)][S_0 - \Gamma_1(n)]^{-1}[S_* - \Gamma_1(n)] \right\} \\ + f \left\{ \frac{\partial^2 S_*}{\partial y^2} - 2 \frac{\partial S_*}{\partial y} [S_* - \Gamma_1(n)]^{-1} \frac{\partial S_*}{\partial y} \right\}.$$

Riccati type PDE

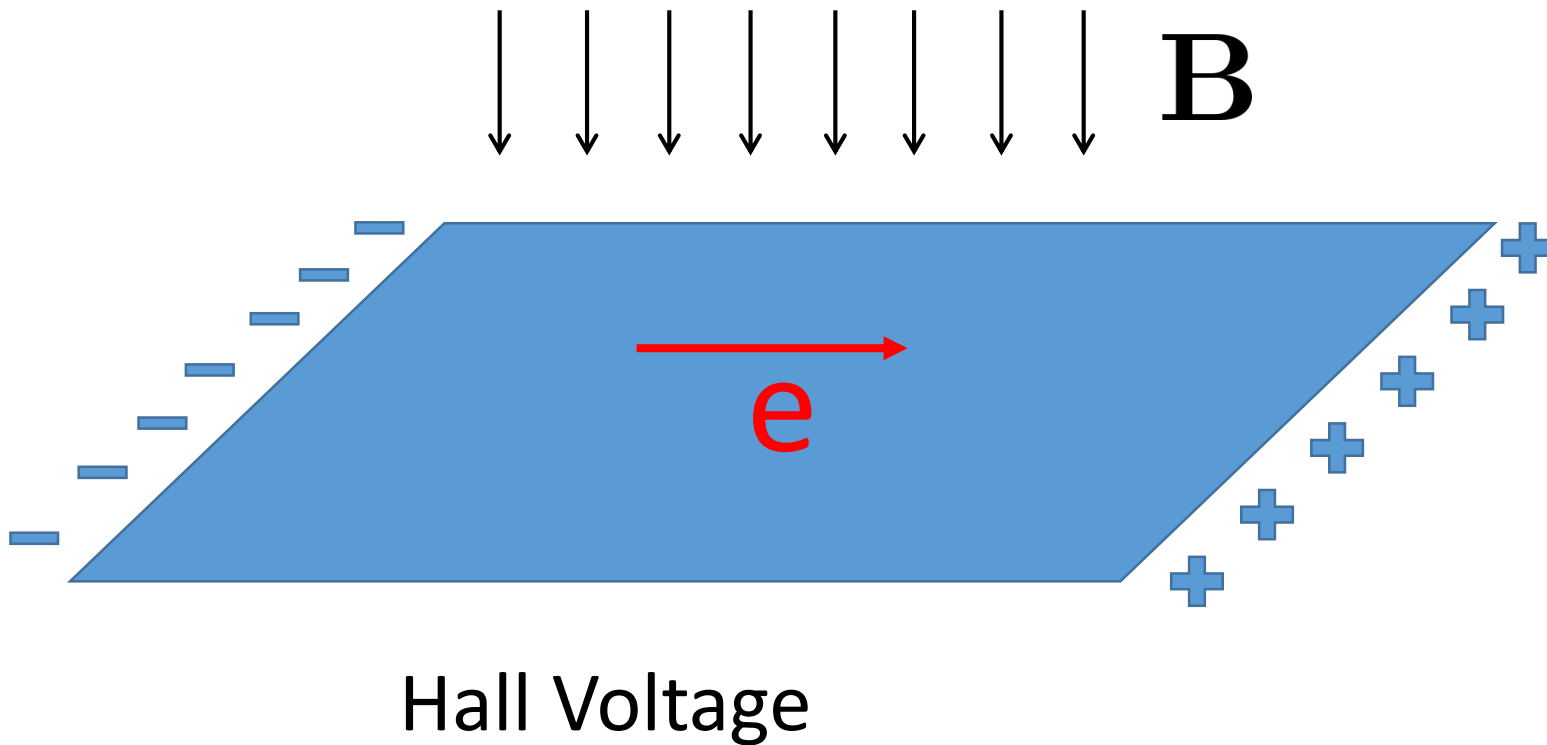
What about 3d?

For a uniform applied field
the local field can vanish
between the torii, even
at finite conductivity ratios



It's constantly a surprise to find what properties a composite can exhibit.

One interesting example:



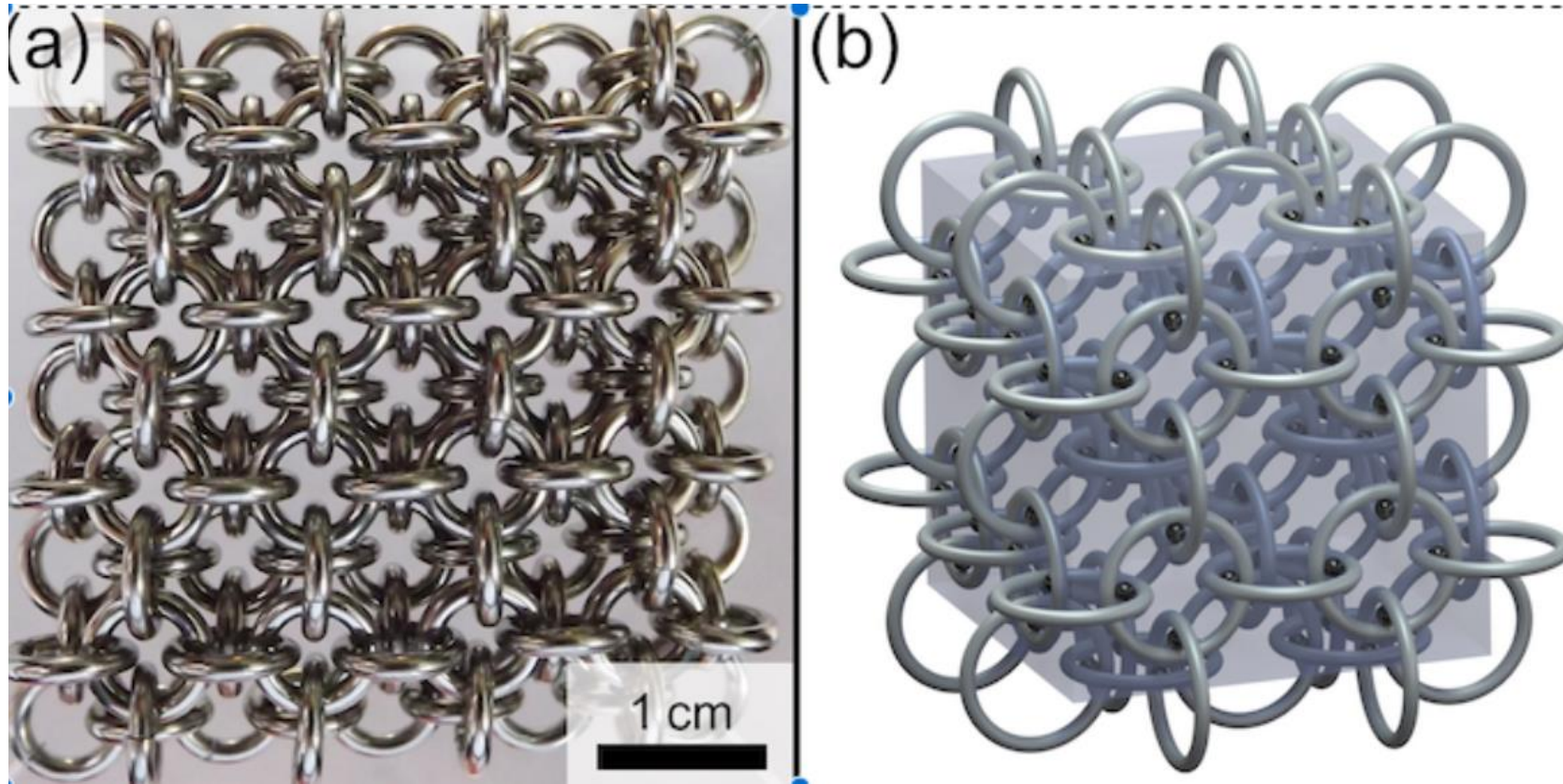
$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Non-symmetric conductivity matrix with the antisymmetric part proportional to \mathbf{B}

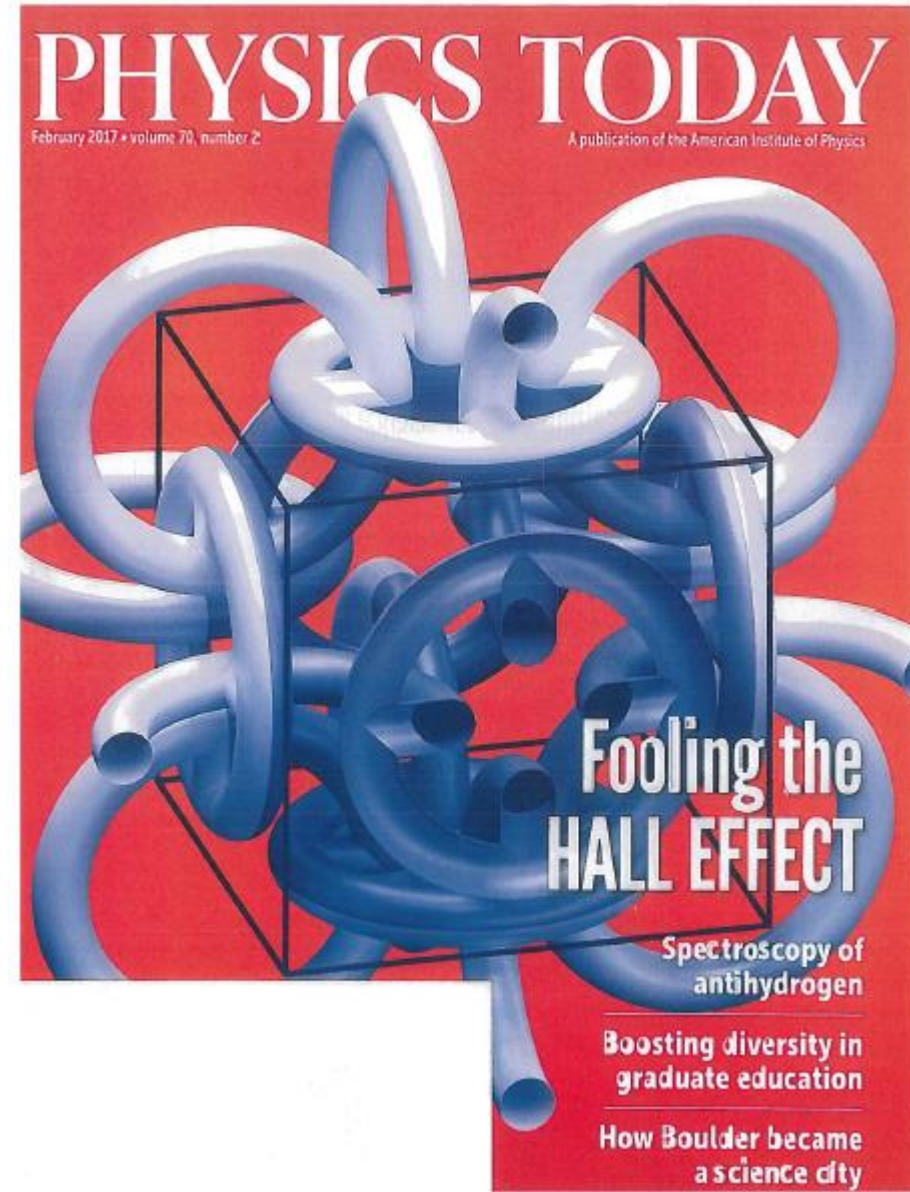
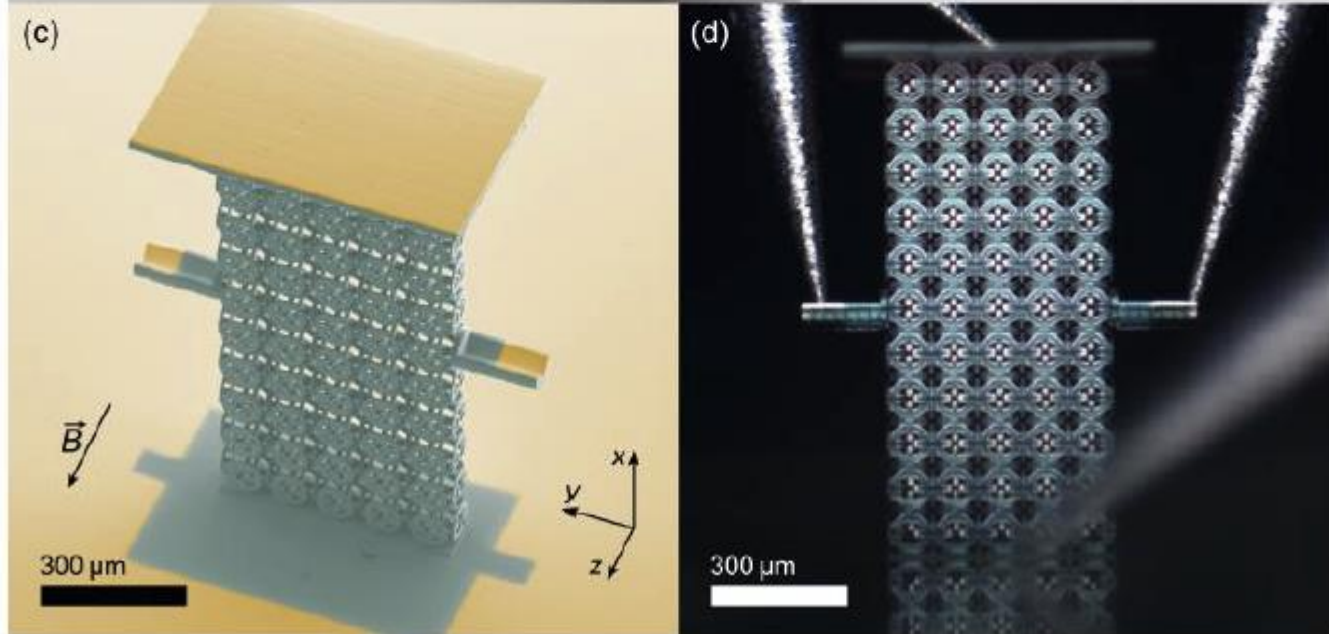
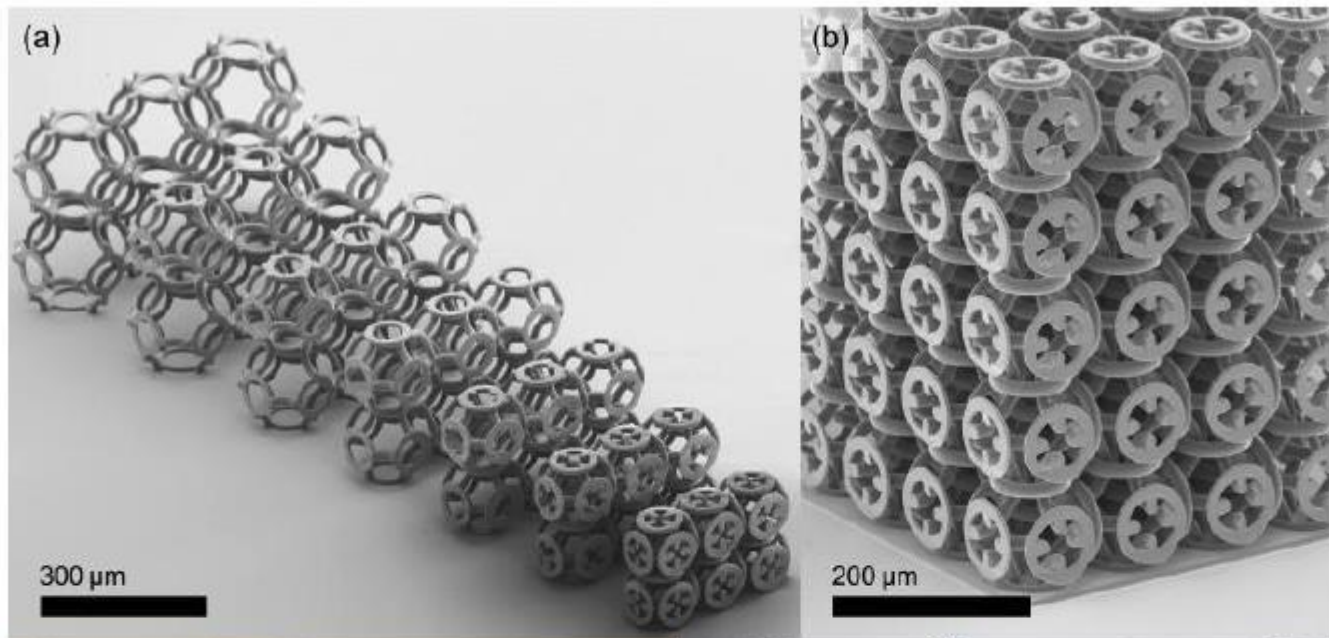
In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

However there is a counterexample!

Geometry suggested by artist Dylan Whyte



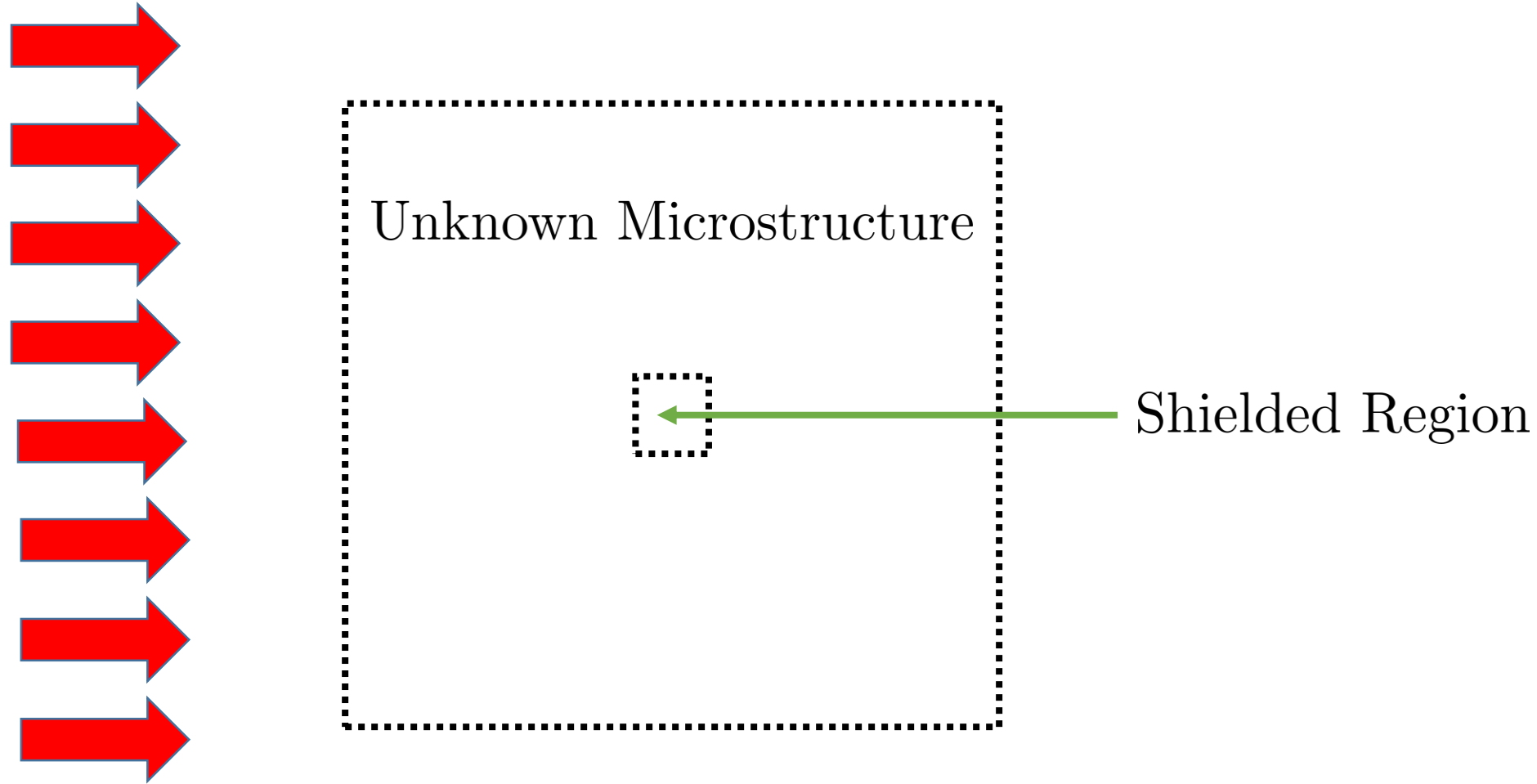
A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)



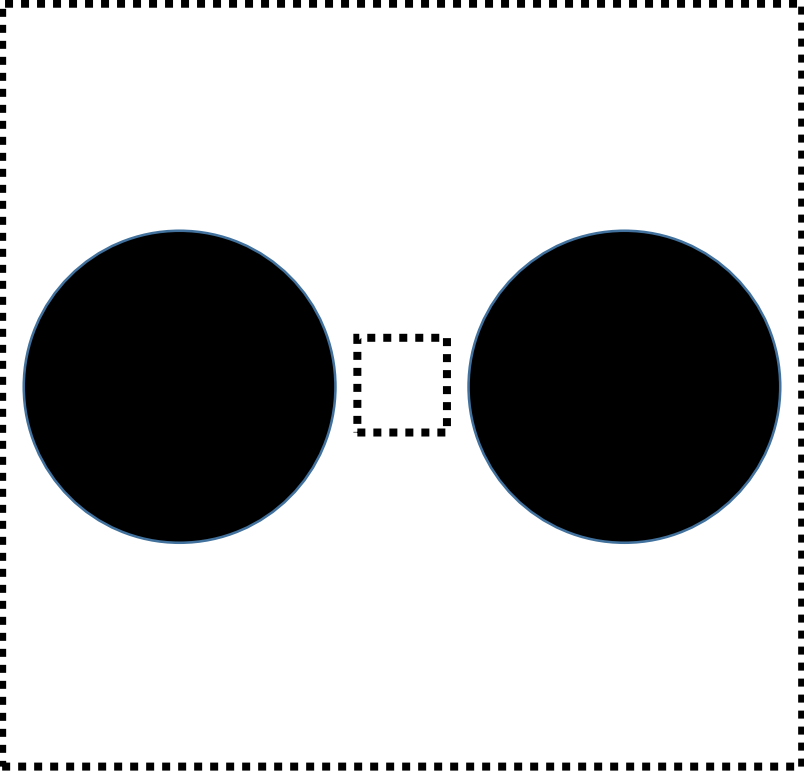
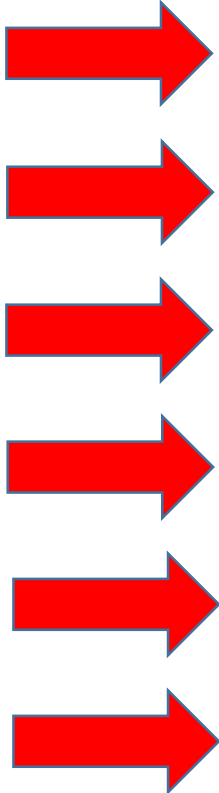
Experimental Realization of Kern, Kadic, Wegener

Back to the shielding problem:

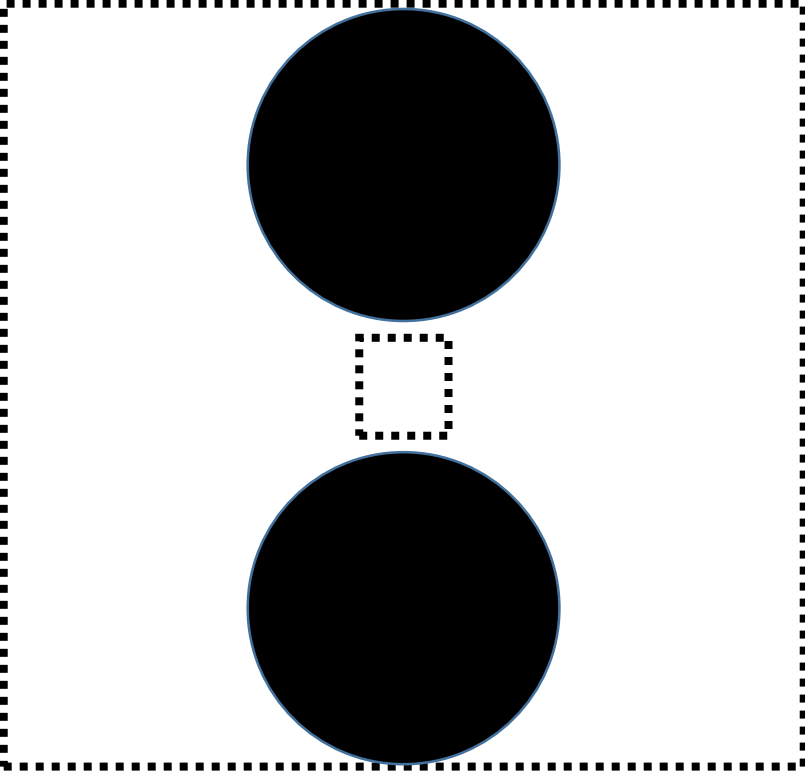
It seems more reasonable to require that there is no microstructure in the shielded region and that the microstructure is localized in a box.



Using Disks:

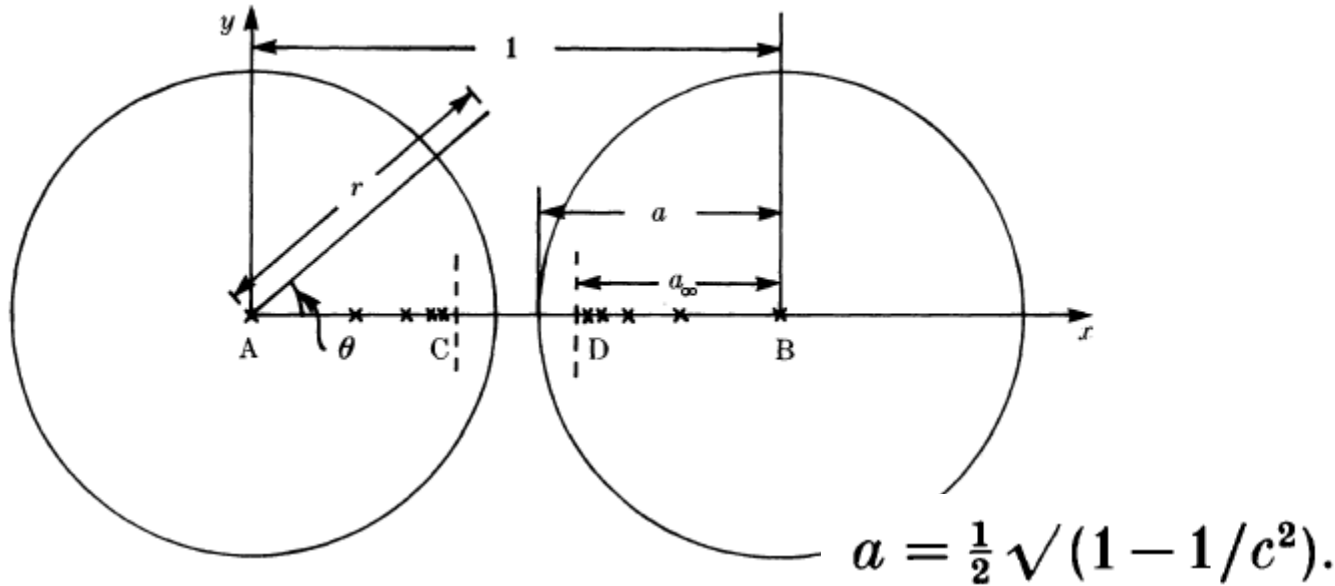


Concentration



Shielding

Field between two highly conducting disks close to touching



McPhedran, Poladian, GWM (1988)

$$B_1 = \frac{-(c/2)(1 - 1/c)}{2s \ln(c) + 1 - 2s[\gamma + \psi(1 + s)]}$$

$$a = \frac{1}{2} \sqrt{1 - 1/c^2}. \quad a_\infty = \frac{1}{2}(1 - 1/c).$$

ψ : Psi or Digamma function

Rigorous Analysis: Lim and Yu (2015)

$$\rho_-(a^2/x) = -\eta\rho_+(x)$$

$$\eta = (\sigma - 1)/(\sigma + 1).$$

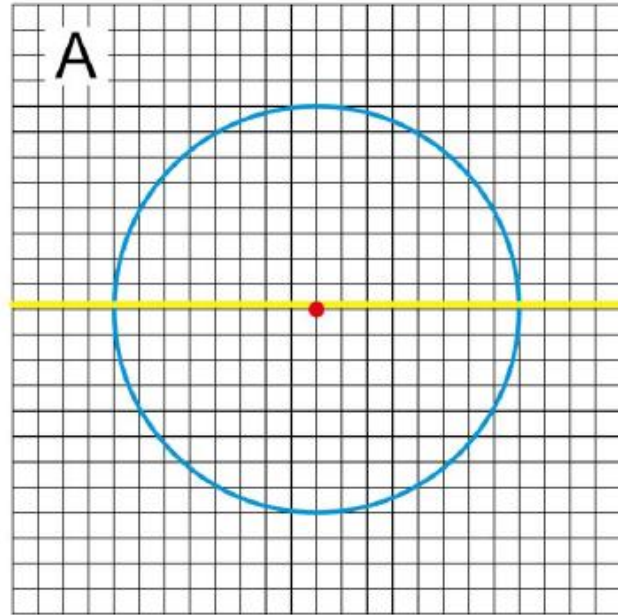
$$\rho_+(1 - x) = -\rho_-(x),$$

$$\rho_-[a^2/(1 - x)] = \eta\rho_-(x)$$

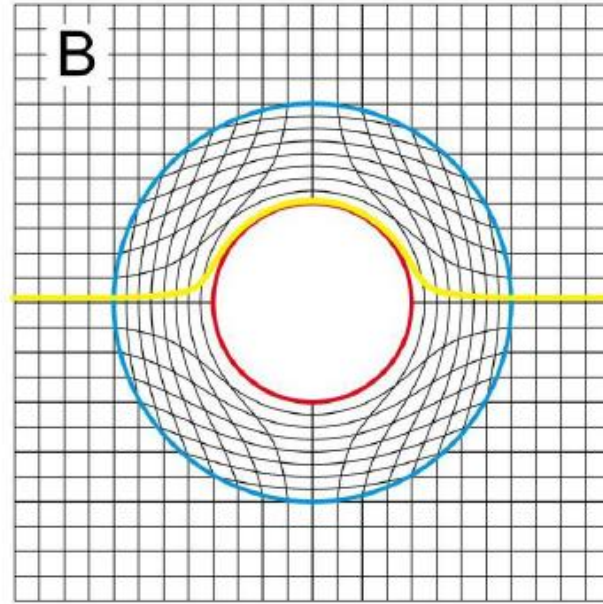
$$\rho_-(x) = A[(a_\infty - x)/(1 - a_\infty - x)]^s$$

$$s = \ln(\eta)/\ln[a_\infty/(1 - a_\infty)]$$

Could use the transformation based approach of Greenleaf, Lassas, and Uhlmann



Stretching space

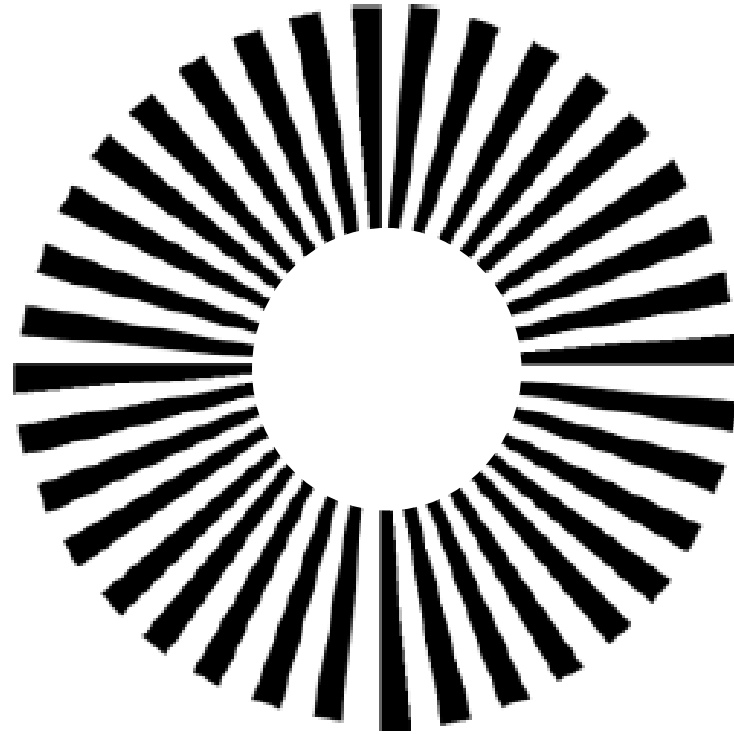
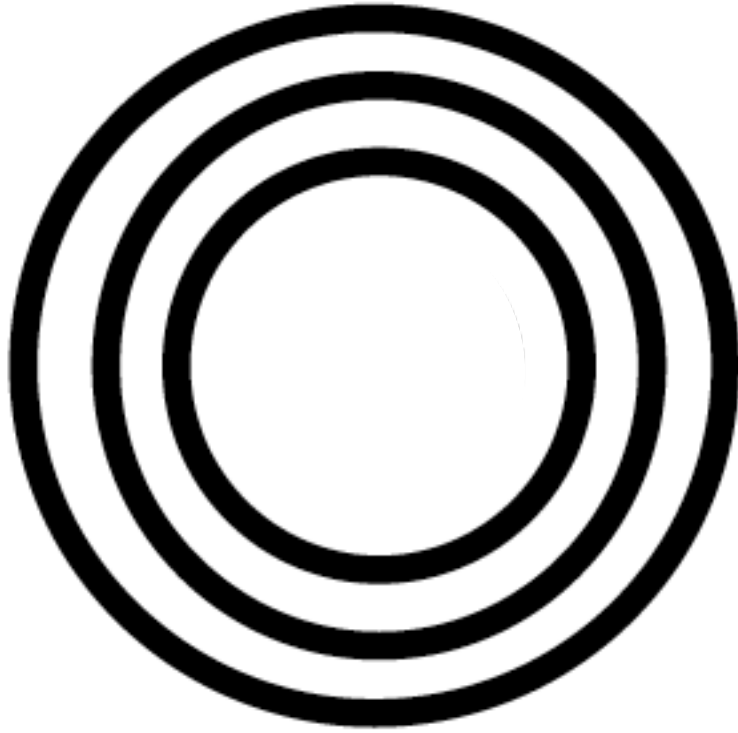


(From Ulf Leonhardt)

Advantages: Works for any external field and creates no disturbance

Disadvantages: Requires extreme conductivities, and if one truncates the solution there is no reason to expect it is optimal.

Or Maybe?



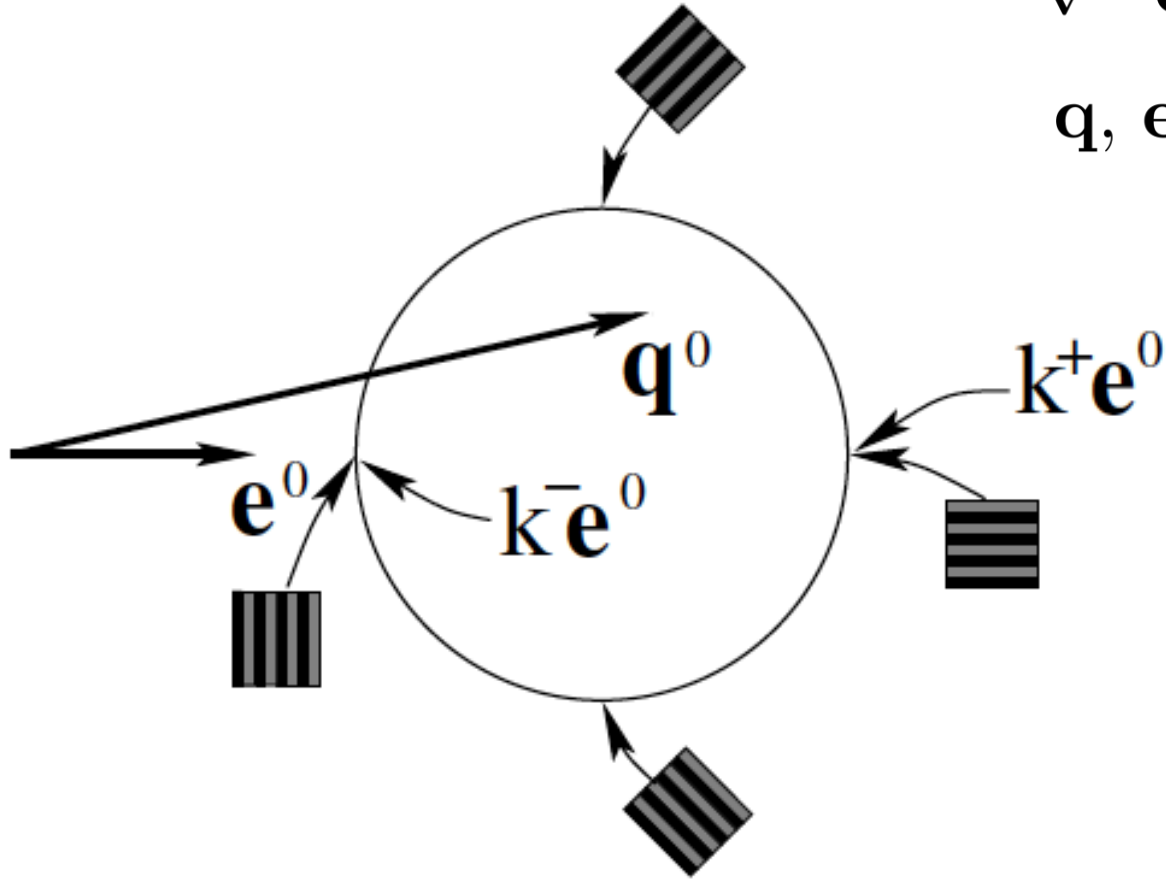
Seems like we are just guessing. Is there a more systematic approach, at least in the case where we use just 2 conducting materials, and we are seeking shielding or concentration for just one applied field?

Possible (average heat current, \mathbf{q}^0 , average temperature gradient, \mathbf{e}^0) pairs in a two phase conducting composite (Raitum, 1978).

$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q}(\mathbf{x}) = k(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \mathbf{e} = -\nabla T$$

$$\mathbf{q}, \mathbf{e} \text{ periodic, } \langle \mathbf{q} \rangle = \mathbf{q}^0, \quad \langle \mathbf{e} \rangle = \mathbf{e}^0,$$

Follows from the Wiener bounds:



$$k^- \mathbf{I} \leq \mathbf{k}^* \leq k^+ \mathbf{I}$$

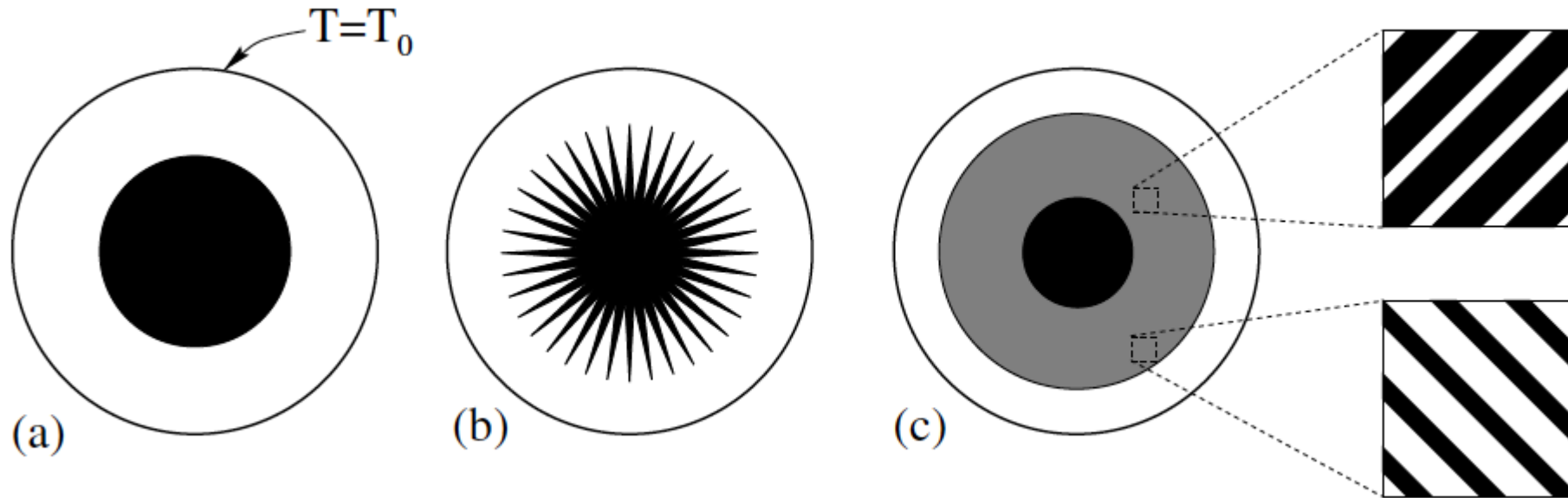
$$k^+ = f k_1 + (1 - f) k_2$$

$$k^- = (f/k_1 + (1 - f)/k_2)^{-1}$$

Solution of the "weak G-closure" problem for conductivity

A model optimization problem:

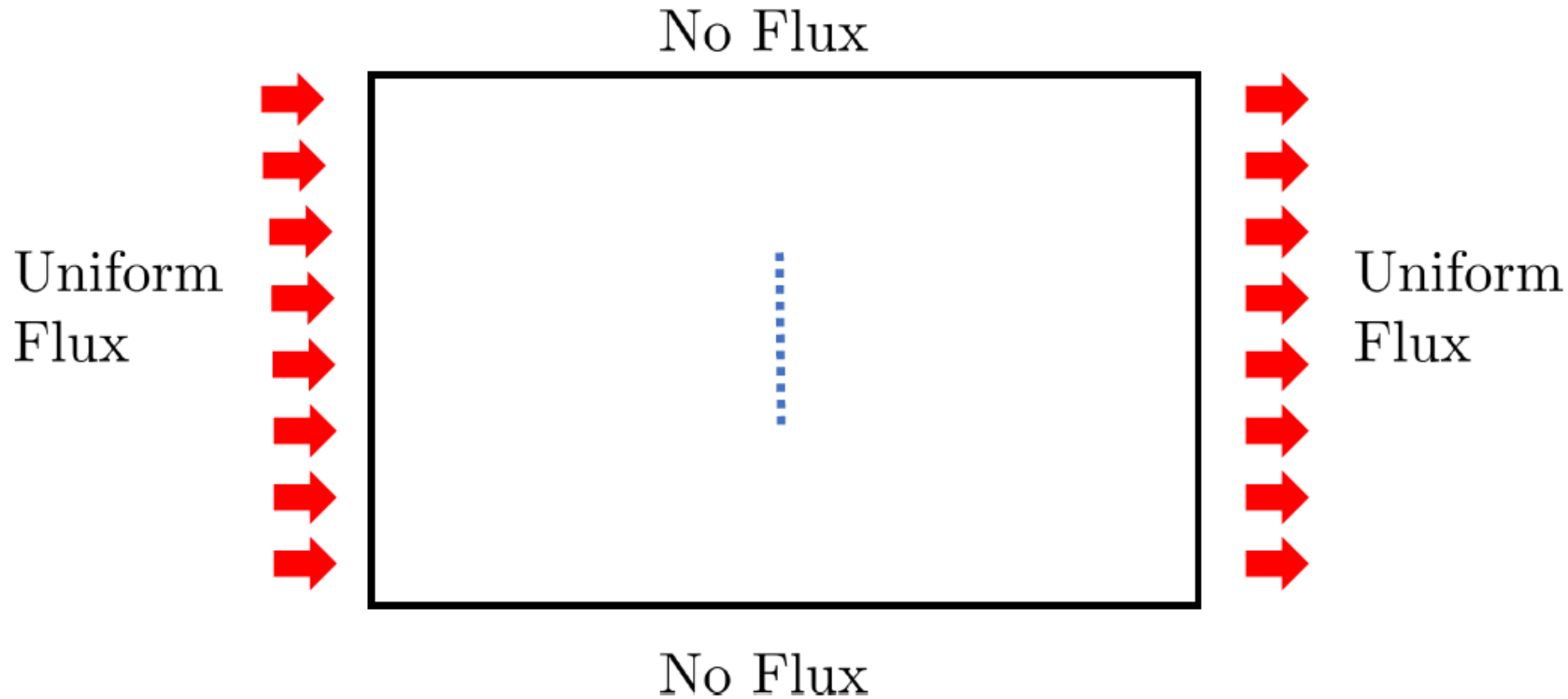
$$-\nabla \cdot \sigma(\mathbf{x}) \nabla T = 1, \quad \sigma = \sigma_1 \text{ or } \sigma_2$$



Solution minimizes $\int_{\Omega} T(\mathbf{x}) d\mathbf{x}$, given fixed amounts of the two materials.

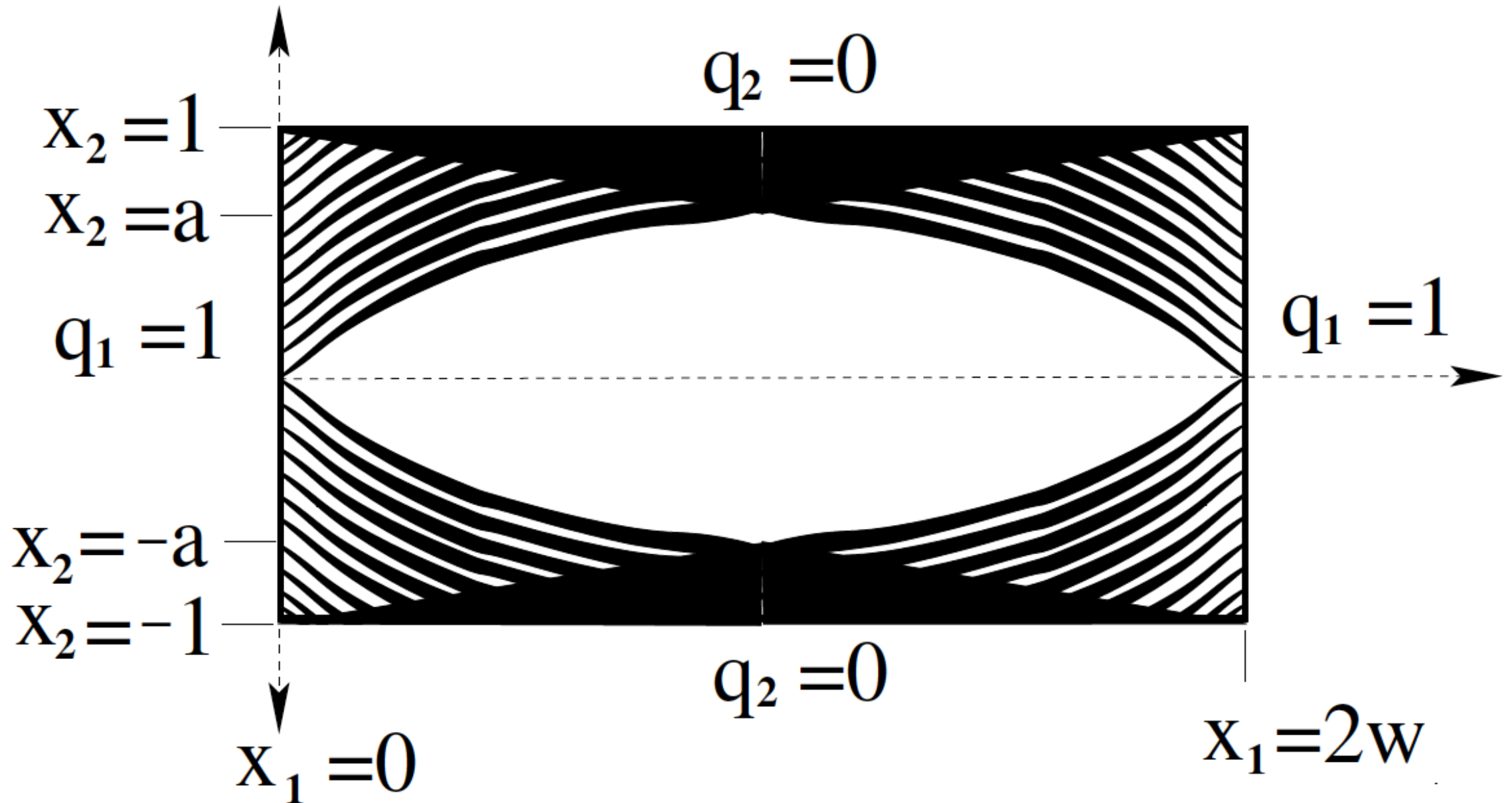
The heat lens problem: Gibiansky, Lurie and Cherkhaev (1988)

Aim: Shield or concentrate flux in the blue dashed interval

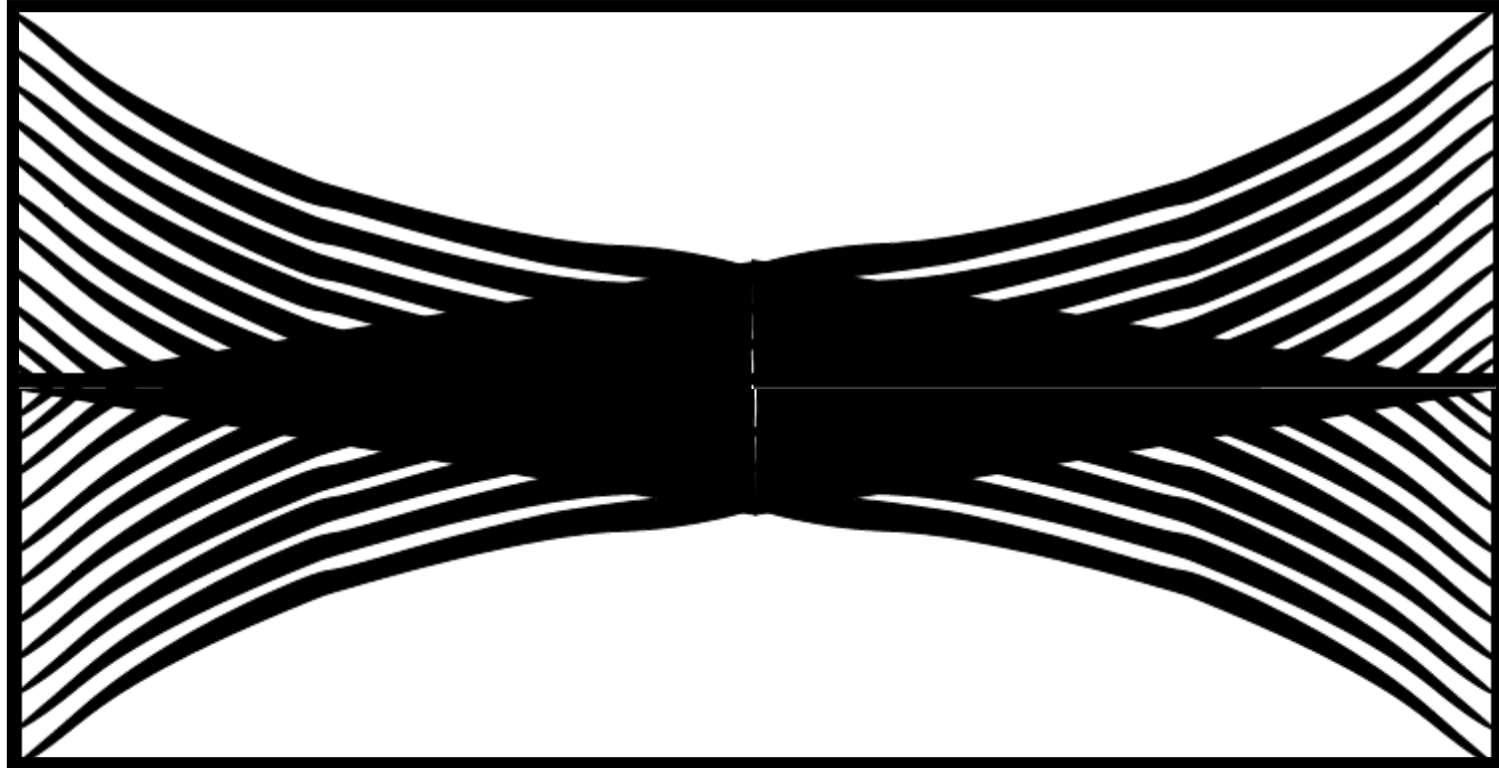


How does one optimally distribute a poor and good conductor to do this?

Field Shield: (Black, good conductor)



Field Concentrator:



What if $k_2 = 0$?

Given \mathbf{q}^0 the weak G-closure provides a linear constraint on \mathbf{e}^0 :

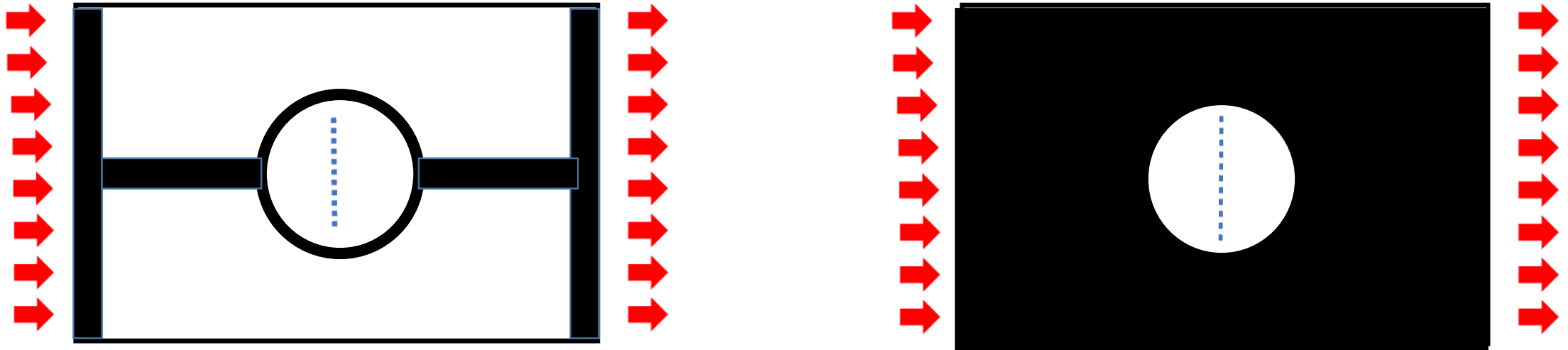
$$\mathbf{q}^0 \cdot \mathbf{q}^0 / (f_1 k_1) \leq \mathbf{q}^0 \cdot \mathbf{e}^0$$

It is attained for laminate geometries but also wire geometries where the effective tensor takes the form:

$$\mathbf{k}^* = f_1 k_1 \mathbf{a} \otimes \mathbf{a}, \quad \mathbf{a} \cdot \mathbf{a} = 1$$

Makes sense: wires are best for conducting current

Many Solutions to the shielding problem:



The weak G-closure is still needed if we:

- (1) Want to minimize the thermal resistance.
- (2) Not use too much of the highly conducting phase (may, e.g., be expensive or heavy).

To solve similar optimization problems for elasticity, can we find the “weak G-Closure” for 3d-elasticity?

At least in the case for 3d printed materials when one phase is void and the other elastically isotropic?

A difficult problem: need to characterize possible (average strain ϵ^0 , average stress σ^0) pairs,

Can assume σ^0 is diagonal and normalized : 2 parameters
Then ϵ^0 has 6 parameters.

So the “weak G-Closure” is described by a set in an 8-dimensional space, 11 if one includes the volume fraction, and bulk and shear moduli of the initial elastic material.

Problem:

$\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\epsilon}(\mathbf{x})$ periodic,

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2.$$

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_1\chi(\mathbf{x}) + \mathbf{C}_2(1 - \chi(\mathbf{x})), \quad \boldsymbol{\sigma}^0 = \langle \boldsymbol{\sigma} \rangle, \quad \boldsymbol{\epsilon}^0 = \langle \boldsymbol{\epsilon} \rangle, \quad f = \langle \chi \rangle$$

Given f what is the range of values the pairs $(\boldsymbol{\sigma}^0, \boldsymbol{\epsilon}^0)$ take in the limit $\mathbf{C}_2 \rightarrow 0$ as the microgeometry varies $\chi(\mathbf{x})$ varies over all possible configurations?

One constraint is immediately implied by sharp bounds on the compliance energy:

$$W_f(\boldsymbol{\sigma}^0) \leq \boldsymbol{\sigma}^0 : \boldsymbol{\epsilon}^0, \quad (*)$$

Explicit expression for $W_f(\boldsymbol{\sigma}_0)$ given by Gibiansky and Cherkaev (1987) and Allaire (1994). Note $W_f(c\mathbf{A}) = c^2 W_f(\mathbf{A})$

Our main result is that these optimal bounds on the compliance tensor also provide optimal bounds on $(\boldsymbol{\epsilon}^0, \boldsymbol{\sigma}^0)$ -pairs. Given $\boldsymbol{\sigma}^0$ they constrain $\boldsymbol{\epsilon}^0$ to lie on one-side of a hyperplane.

$$W_f(\boldsymbol{\sigma}^0) = \boldsymbol{\sigma}^0 : \mathbf{C}_1^{-1} \boldsymbol{\sigma}^0 + \frac{f}{2\mu} g(\mathbf{C}_1, \boldsymbol{\sigma}^0), \quad (\text{Using Allaire's notation.})$$

Suppose the stress has eigenvalues σ_1 , σ_2 and σ_3 . Can assume at most one eigenvalue is negative, and $\sigma_1 \leq \sigma_2 \leq \sigma_3$. When all are non-negative, and $\lambda > 0$:

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \leq \sigma_1 + \sigma_2, \\ &= (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \geq \sigma_1 + \sigma_2, \end{aligned}$$

while when one eigenvalue, namely σ_1 , is negative,

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)} \left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda} \sigma_1 \right)^2 \\ &\text{if } \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1 \text{ and } \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1, \\ &= (\sigma_3 + \sigma_2)^2 + \sigma_1^2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 + \sigma_2 \leq \frac{-\mu}{\mu + \lambda} \sigma_1, \\ &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{2\mu}{\mu + \lambda} \sigma_1 \sigma_2 - \frac{\lambda}{2\mu + 3\lambda} (\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 - \sigma_2 \geq \frac{-\mu}{\mu + \lambda} \sigma_1. \end{aligned}$$

The required geometries are pentmodes, materials with elastic tensor

$$\mathbf{C}^* = \alpha \mathbf{A} \otimes \mathbf{A}, \quad \mathbf{A} : \mathbf{A} = 1$$

that are optimal in the sense that

$$\alpha = 1/W_f(\mathbf{A})$$

Given any $\boldsymbol{\sigma}_0$ and $\boldsymbol{\epsilon}_0$ so that (*) holds as an equality, we choose

$$\mathbf{A} = \boldsymbol{\sigma}_0 / \sqrt{\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0}$$

and then

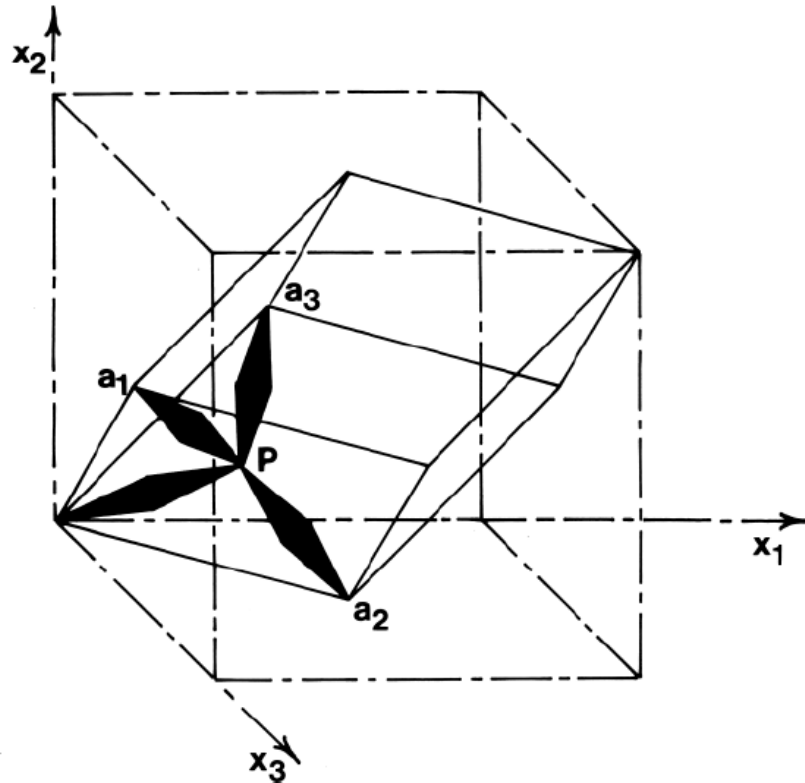
$$\mathbf{C}^* \boldsymbol{\epsilon}_0 = \alpha \boldsymbol{\sigma}_0 W_f(\boldsymbol{\sigma}_0) / (\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0) = \alpha \boldsymbol{\sigma}_0 W_f(\mathbf{A}) = \boldsymbol{\sigma}_0$$

as desired.

What are pentamodes?

New classes of elastic materials (with Cherkaev, 1995)

A three dimensional pentamode material which can support any prescribed loading



Like a fluid it only supports one loading, unlike a fluid that loading may be anisotropic

Pentamode structures are a sort of anisotropic inhomogeneous fluid

$$\mathbf{C}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \otimes \mathbf{A}(\mathbf{x}), \quad \nabla \cdot \mathbf{A} = 0,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2$$

have the solution

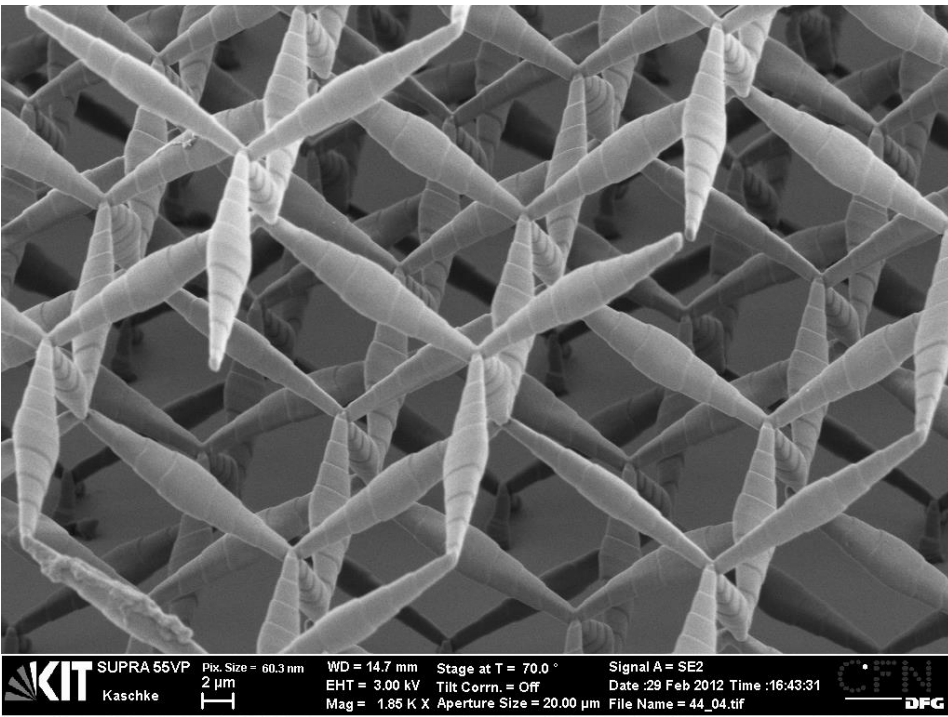
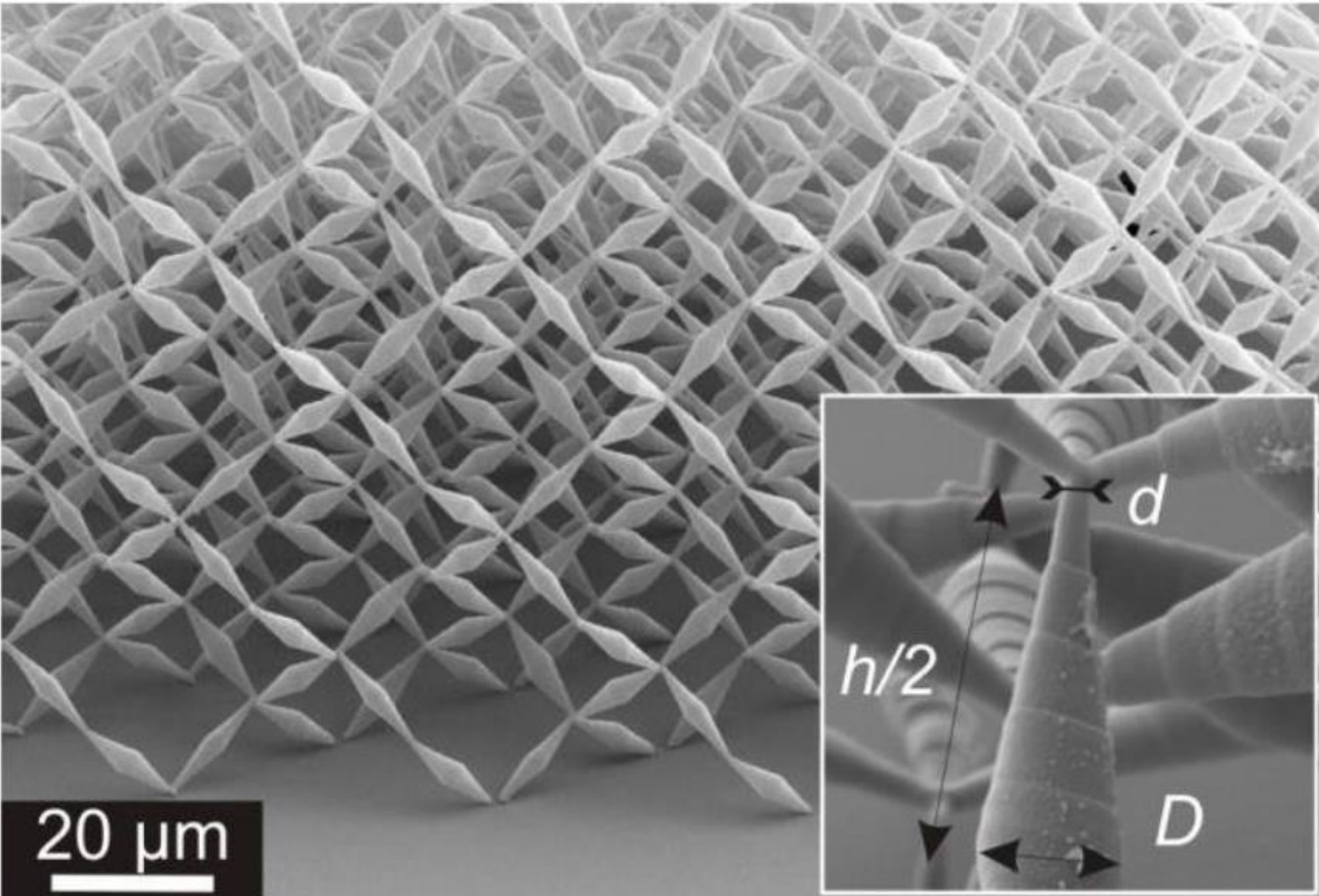
$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha \mathbf{A}(\mathbf{x})$$

where $\alpha =$ "a constant" is the analog of pressure, and

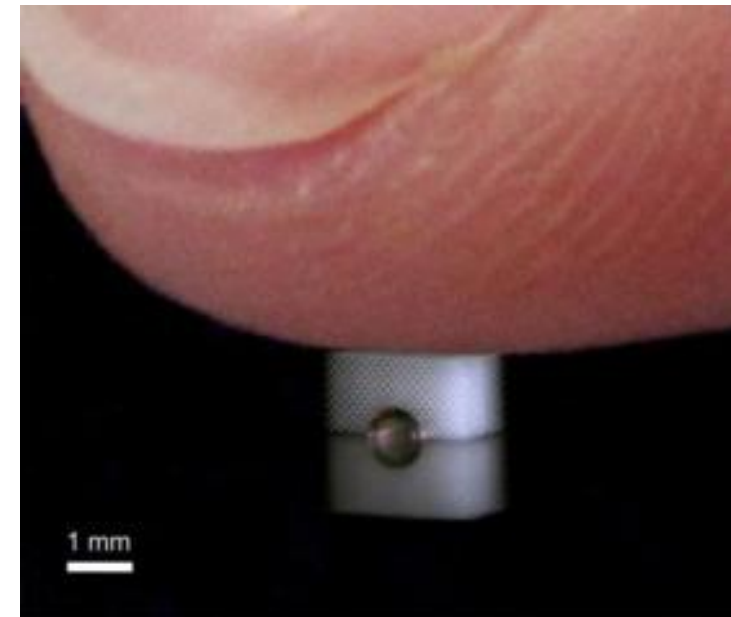
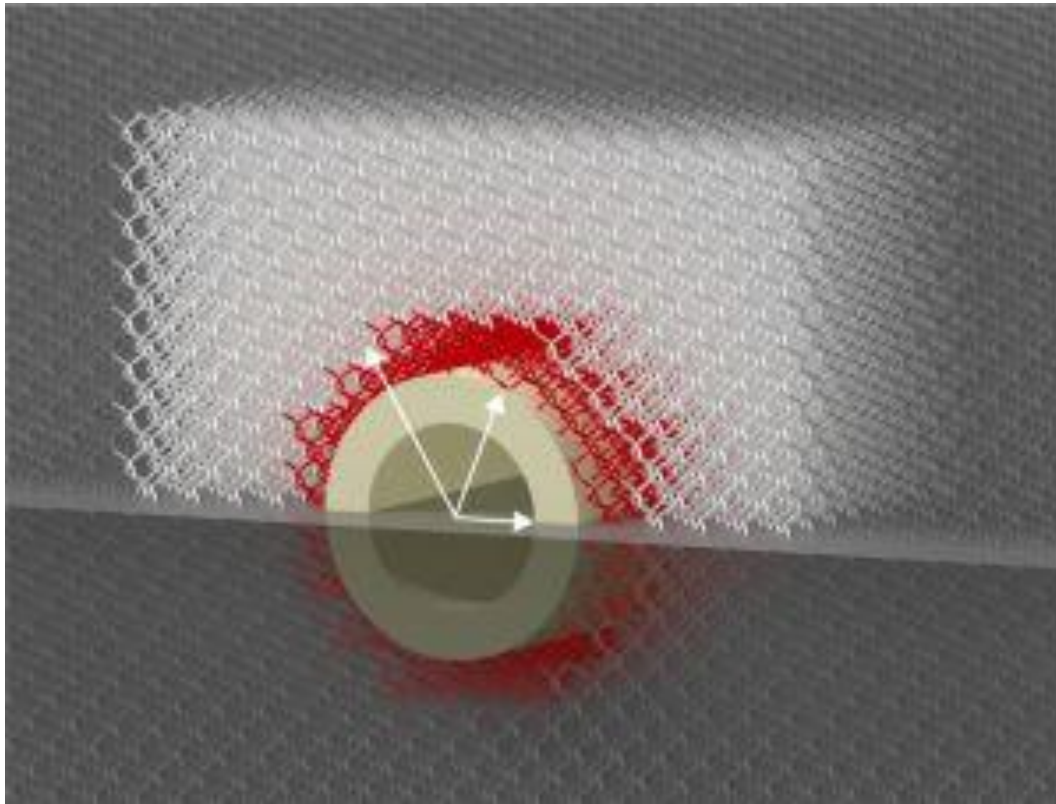
$$\alpha = \text{Tr}[\mathbf{A}(\mathbf{x})\nabla \mathbf{u}],$$

constrains $\nabla \mathbf{u}$. Thus $\mathbf{A}(\mathbf{x})$ is a sort of anisotropic "compressibility"

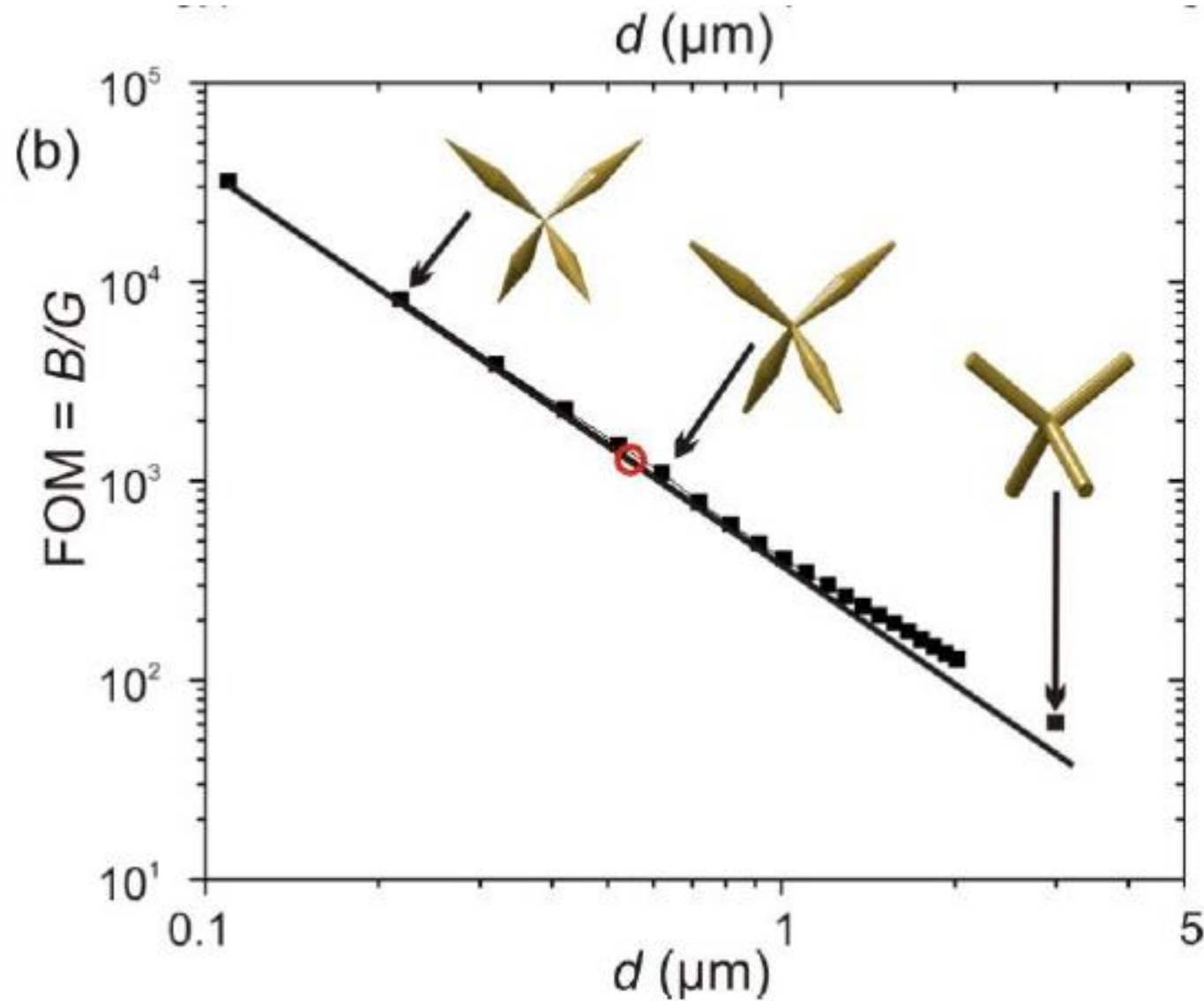
Realization of Kadic et.al. 2012



Cloak making an object “unfeelable”:
Buckmann et. al. (2014)

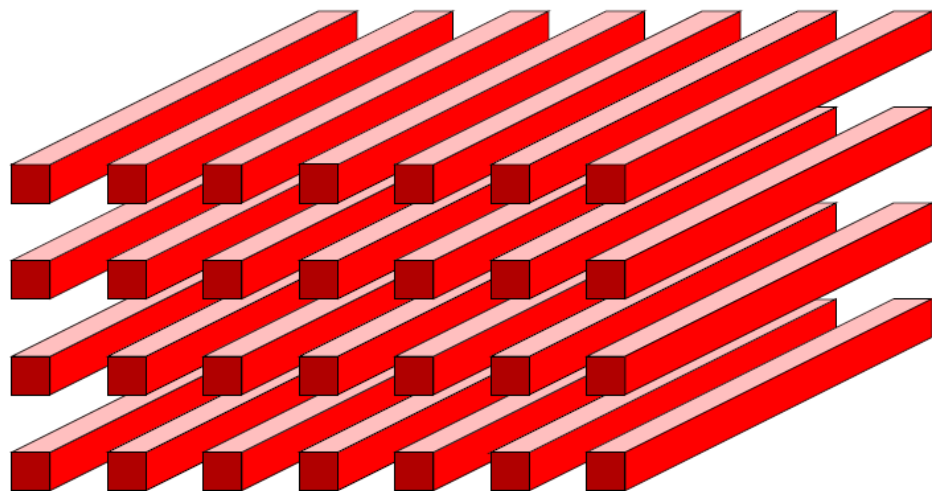
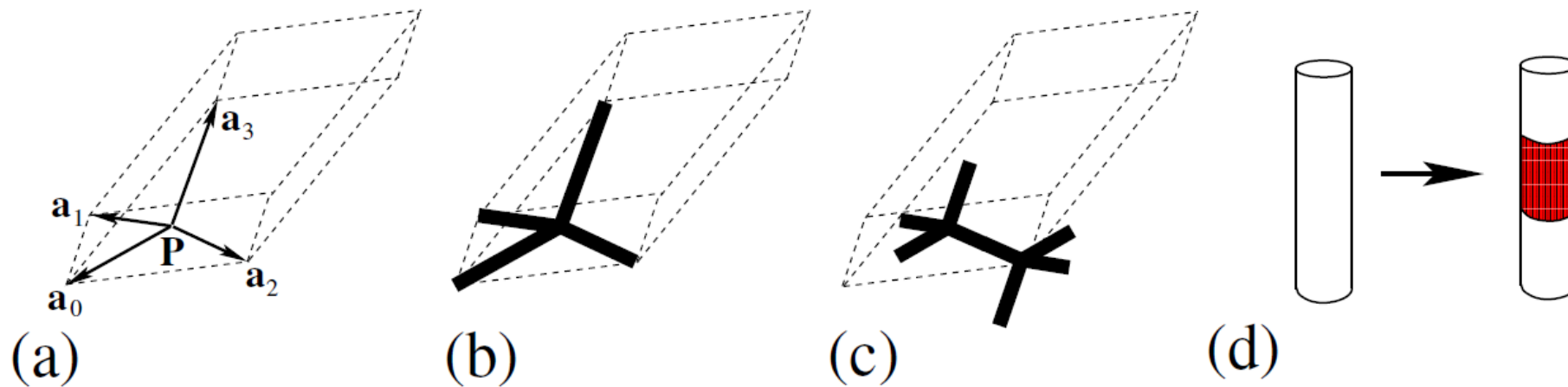


Kadic. et.al 2012

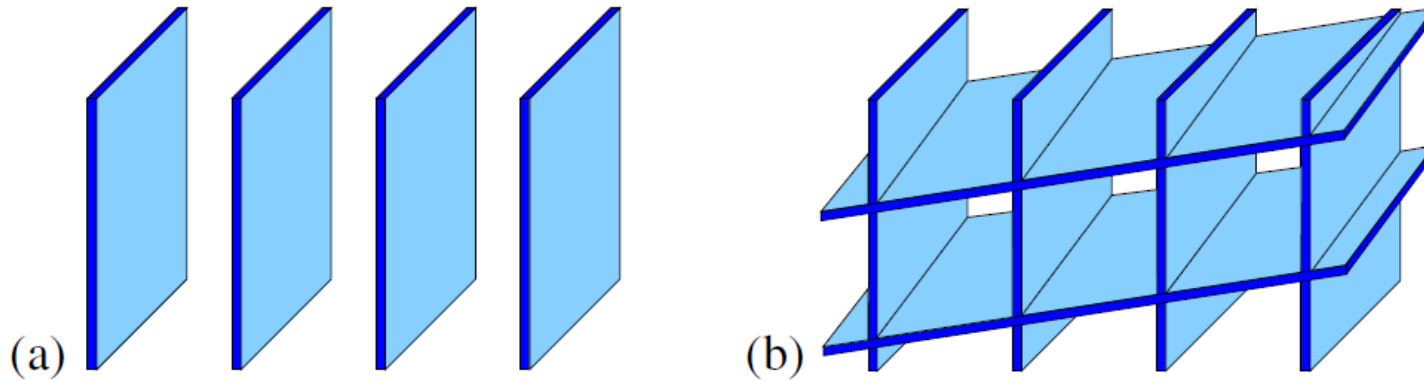


Disadvantage: not only does the shear modulus go to zero as they are made more ideal, but also the bulk modulus goes to zero

Modifying the pentamodes:



Idea of proof: Insert into the material attaining the energy bounds a thin walled structure with sets of parallel walls:



Inside the walls put the appropriate modified pentamode material. Thus we obtain an optimal pentamode attaining the energy bounds.

For elastically isotropic materials one has the Hashin-Shtrikman Bounds

$$\kappa_* \geq f_1 \kappa_1 + f_2 \kappa_2 - \frac{f_1 f_2 (\kappa_1 - \kappa_2)^2}{f_2 \kappa_1 + f_1 \kappa_2 + 4\mu_2/3},$$

$$\mu_* \geq f_1 \mu_1 + f_2 \mu_2 - \frac{f_1 f_2 (\mu_1 - \mu_2)^2}{f_2 \mu_1 + f_1 \mu_2 + \mu_2 (9\kappa_2 + 8\mu_2) / [6(\kappa_2 + 2\mu_2)]}$$

$$\kappa_* \leq f_1 \kappa_1 + f_2 \kappa_2 - \frac{f_1 f_2 (\kappa_1 - \kappa_2)^2}{f_2 \kappa_1 + f_1 \kappa_2 + 4\mu_1/3},$$

$$\mu_* \leq f_1 \mu_1 + f_2 \mu_2 - \frac{f_1 f_2 (\mu_1 - \mu_2)^2}{f_2 \mu_1 + f_1 \mu_2 + \mu_1 (9\kappa_1 + 8\mu_1) / [6(\kappa_1 + 2\mu_1)]}$$

The optimal pentamode supporting hydrostatic stress $\boldsymbol{\sigma}^0 = \mathbf{I}$, is a material that for fixed $f_1 = 1 - f_2$ in the limit $\kappa_2, \mu_2 \rightarrow 0$ attains the bulk modulus upper bound, yet has zero shear modulus, $\mu_* = 0$.

We can go much further and go a long way to completely characterizing the G-closure of 3d (and 2d) printed materials.

Joint work with Marc Briane and Davit Harutyunyan

Problem:

$\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\epsilon}(\mathbf{x})$ periodic,

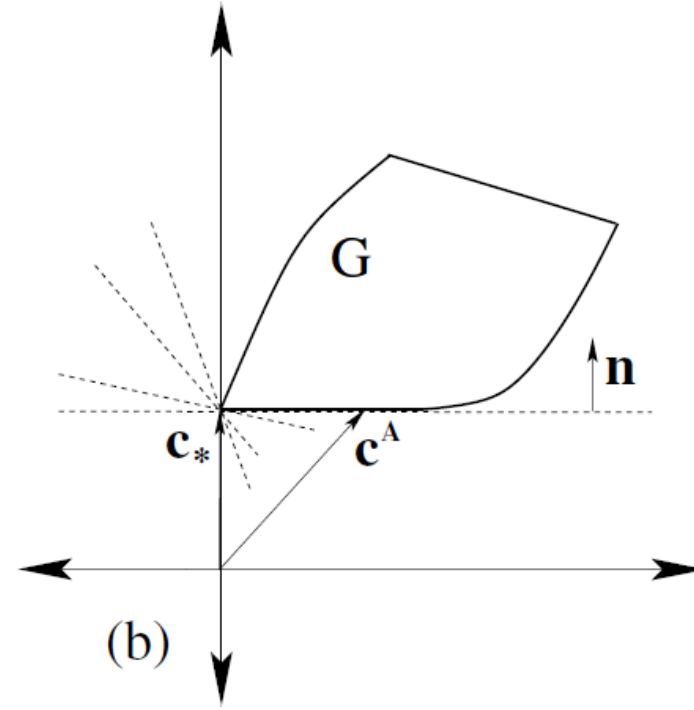
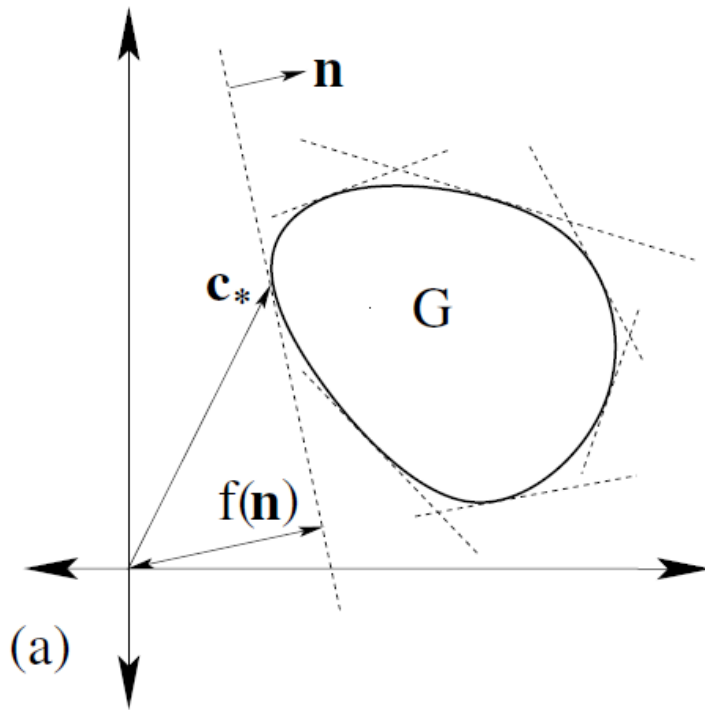
$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2.$$

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_1\chi(\mathbf{x}) + \mathbf{C}_2(1 - \chi(\mathbf{x})), \quad \boldsymbol{\sigma}^0 = \langle \boldsymbol{\sigma} \rangle, \quad \boldsymbol{\epsilon}^0 = \langle \boldsymbol{\epsilon} \rangle, \quad f = \langle \chi \rangle$$

By linearity $\boldsymbol{\sigma}^0 = \mathbf{C}^*\boldsymbol{\epsilon}^0$. Given f what is the range of values the effective tensor \mathbf{C}^* takes in the limit $\mathbf{C}_2 \rightarrow 0$ as the microgeometry varies $\chi(\mathbf{x})$ varies over all possible configurations?

Recall: A convex set G can be characterized by its Legendre transform:

$$f(\mathbf{n}) = \min_{\mathbf{c} \in G} \mathbf{n} \cdot \mathbf{c}.$$



G-closures are not convex sets but can be characterized by their W-transform

$$W_f(\mathbf{N}, \mathbf{N}') = \min_{\mathbf{C}_* \in GU_f} (\mathbf{C}_*, \mathbf{N}) + (\mathbf{C}_*^{-1}, \mathbf{N}'),$$

$$(\mathbf{N}, \mathbf{C}) = N_{ijkl} C_{ijkl}$$

$$\bigcap_{\substack{\mathbf{N}, \mathbf{N}' \geq 0 \\ \mathbf{N}\mathbf{N}' = 0}} \{\mathbf{C} : (\mathbf{C}, \mathbf{N}) + (\mathbf{C}^{-1}, \mathbf{N}') \geq W_f(\mathbf{N}, \mathbf{N}')\} = GU_f.$$

W-transforms generalize the idea of Legendre transforms

$$\mathbf{N} = \sum_{i=1}^2 \epsilon_i^0 \otimes \epsilon_i^0, \quad \mathbf{N}' = \sum_{j=1}^4 \sigma_j^0 \otimes \sigma_j^0,$$

Need to know the 7 energy functions

$$W_f^0(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \sigma_6^0) = \min_{C_* \in GU_f} \sum_{j=1}^6 \sigma_j^0 : C_*^{-1} \sigma_j^0,$$

$$W_f^1(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \epsilon_1^0) = \min_{C_* \in GU_f} \left[\epsilon_1^0 : C_* \epsilon_1^0 + \sum_{j=1}^5 \sigma_j^0 : C_*^{-1} \sigma_j^0 \right],$$

$$W_f^2(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \epsilon_1^0, \epsilon_2^0) = \min_{C_* \in GU_f} \left[\sum_{i=1}^2 \epsilon_i^0 : C_* \epsilon_i^0 + \sum_{j=1}^4 \sigma_j^0 : C_*^{-1} \sigma_j^0 \right],$$

$$W_f^3(\sigma_1^0, \sigma_2^0, \sigma_3^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0) = \min_{C_* \in GU_f} \left[\sum_{i=1}^3 \epsilon_i^0 : C_* \epsilon_i^0 + \sum_{j=1}^3 \sigma_j^0 : C_*^{-1} \sigma_j^0 \right],$$

$$W_f^4(\sigma_1^0, \sigma_2^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0) = \min_{C_* \in GU_f} \left[\sum_{i=1}^4 \epsilon_i^0 : C_* \epsilon_i^0 + \sum_{j=1}^2 \sigma_j^0 : C_*^{-1} \sigma_j^0 \right],$$

$$W_f^5(\sigma_1^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0) = \min_{C_* \in GU_f} \left[\left(\sum_{i=1}^5 \epsilon_i^0 : C_* \epsilon_i^0 \right) + \sigma_1^0 : C_*^{-1} \sigma_1^0 \right],$$

$$W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0) = \min_{C_* \in GU_f} \sum_{i=1}^6 \epsilon_i^0 : C_* \epsilon_i^0.$$

Orthogonality conditions

$$(\epsilon_i^0, \sigma_j^0) = 0, \quad (\epsilon_i^0, \epsilon_k^0) = 0, \quad (\sigma_j^0, \sigma_\ell^0) = 0$$

for all i, j, k, ℓ with $i \neq j$, $i \neq k$, $j \neq \ell$.

Result of Avellaneda (1987): If $C_1 \geq C_2$ then

$$W_f^0(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \sigma_6^0) = \min_{C_* \in GU_f} \sum_{j=1}^6 \sigma_j^0 : C_*^{-1} \sigma_j^0,$$

$$W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0) = \min_{C_* \in GU_f} \sum_{i=1}^6 \epsilon_i^0 : C_* \epsilon_i^0.$$

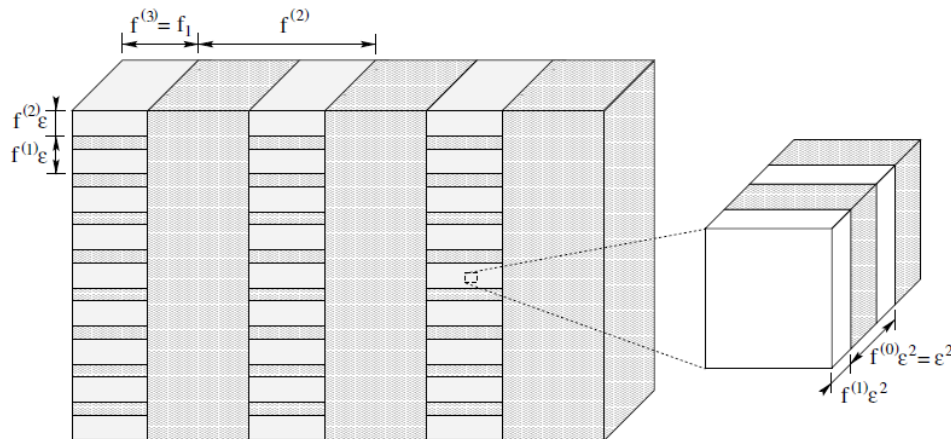
can be computed

They are attained by sequentially layered laminates, and we call the material which attains the minimum in

$$W_f^0(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \sigma_6^0) = \min_{C_* \in GU_f} \sum_{j=1}^6 \sigma_j^0 : C_*^{-1} \sigma_j^0,$$

the Avellaneda material, with elasticity tensor

$$C_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \sigma_6^0)$$



Maxwell (1873)

Obvious bounds:

$$\sum_{j=1}^5 \sigma_j^0 : [\mathbf{C}_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, 0)]^{-1} \sigma_j^0 \leq W_f^1(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \epsilon_1^0),$$

$$\sum_{j=1}^4 \sigma_j^0 : [\mathbf{C}_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, 0, 0)]^{-1} \sigma_j^0 \leq W_f^2(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \epsilon_1^0, \epsilon_2^0),$$

$$\sum_{j=1}^3 \sigma_j^0 : [\mathbf{C}_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, 0, 0, 0)]^{-1} \sigma_j^0 \leq W_f^3(\sigma_1^0, \sigma_2^0, \sigma_3^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0),$$

$$\sum_{j=1}^2 \sigma_j^0 : [\mathbf{C}_f^A(\sigma_1^0, \sigma_2^0, 0, 0, 0, 0)]^{-1} \sigma_j^0 \leq W_f^4(\sigma_1^0, \sigma_2^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0),$$

$$\sigma_1^0 : [\mathbf{C}_f^A(\sigma_1^0, 0, 0, 0, 0, 0)]^{-1} \sigma_1^0 \leq W_f^5(\sigma_1^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0),$$

$$0 \leq W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0).$$

Main result: in many cases these bounds are sharp

Theorem (GWM, Briane, Harutyunyan):

$$\lim_{\delta \rightarrow 0} W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0) = \sum_{j=1}^3 \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0,$$

$$\lim_{\delta \rightarrow 0} W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0) = \sum_{j=1}^2 \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0,$$

$$\lim_{\delta \rightarrow 0} W_f^5(\boldsymbol{\sigma}_1^0, \boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0) = \boldsymbol{\sigma}_1^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0,$$

$$\lim_{\delta \rightarrow 0} W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) = 0.$$

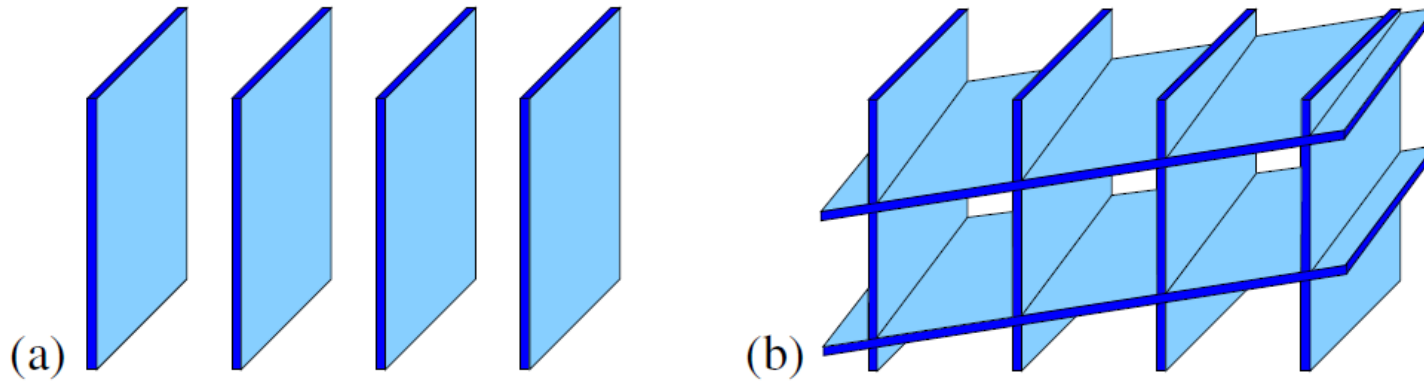
When ϵ_1^0 has one zero eigenvalue, and the other eigenvalues of opposite signs,

$$W_f^1(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, \epsilon_1^0) = \sum_{j=1}^5 \sigma_j^0 : [C_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \sigma_5^0, 0)]^{-1} \sigma_j^0$$

When $\det(\epsilon_1^0 + t\epsilon_2^0) = 0$ has at least two roots and $\epsilon(t) = \epsilon_1^0 + t\epsilon_2^0$ is never positive or negative definite

$$W_f^2(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, \epsilon_1^0, \epsilon_2^0) = \sum_{j=1}^4 \sigma_j^0 : [C_f^A(\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_4^0, 0, 0)]^{-1} \sigma_j^0$$

Idea of proof: Insert into the Avellaneda material a thin walled structure with sets of parallel walls:



Inside the walls put the appropriate multimode material

Thank You!

Extending the Theory of Composites to Other Areas of Science

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