Spectral geometry of the Neumann-Poincaré operator on three dimensional domains

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We introduce the spectral structure of NP operators in three dimensions:

- $\S1\sim\S6$ Weyl's law of Neumann–Poincaré operators (Y. M.)
- $\S7\sim\S8$ Hadamard's variational formulas of Neumann–Poincaré operators (K. Ando, E. Ushikoshi, H. Kang and Y. M.)

§1 Introduction: The electro-static NP operator

- (The electro-static Neumann-Poincaré (NP) operator)

  Let Ω be a $C^{1,\alpha}$ bounded region in $\mathbf{R}^n$ $(n = 2, 3)$.

  The NP operator $\mathcal{K}_{\partial\Omega} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is defined by

$$\mathcal{K}_{\partial\Omega}[\psi](x) \equiv \int_{\partial\Omega} \psi(y) \cdot \nu_y E(x, y)\, dS_y,$$

where

$$E(x, y) = \begin{cases} \frac{1}{2\pi} \log |x - y|, & \text{if } n = 2, \\ \frac{-1}{4\pi} \frac{1}{|x-y|}, & \text{if } n = 3 \end{cases}$$

dS_y is the line or surface element and ν_y is the outer normal derivative on $\partial\Omega$.

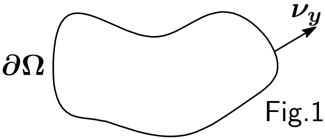


Fig.1

(Problem) Can one hear the shape of $\partial\Omega$ by the eigenvalues of $\mathcal{K}_{\partial\Omega}$?

§1.1 Introduction (A history) (Why spectrum?)

$$\boldsymbol{E(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & n = 2, \\ \frac{-1}{4\pi|x|}, & n = 3 \end{cases}}$$

$\triangle \boldsymbol{E(x) = \delta_0(x)}$ (Dirac's delta), $\triangle \boldsymbol{E(x) = 0 \ (x \neq 0)}$.

Single layer potential: $\boldsymbol{S[\psi](x) = \int_{\partial\Omega} E(x-y)\psi(y)dS_y}$

$$\triangle \boldsymbol{S[\psi](x) = 0 \ (x \notin \partial\Omega)}$$

Classical Neumann Problem:

$$\begin{cases} \triangle \boldsymbol{u = 0} & \text{in } \boldsymbol{\Omega}, \\ \boldsymbol{\frac{\partial}{\partial\nu}u = g} & \text{on } \boldsymbol{\partial\Omega}. \end{cases}$$

Look for solution of the form $\boldsymbol{u(x) = S[\psi](x)}$.
(it is automatically harmonic & should satisfy the boundary condition)

§1.1 Introduction (A history) (Why spectrum?)

How to find the boundary condition

Jump relation of single layer potential:

$$\frac{\partial}{\partial \nu} S[\psi]|_{\pm} = (\pm \frac{1}{2} I + \mathcal{K}^*_{\partial\Omega})[\psi] \quad \text{on } \partial\Omega$$

$$\triangle S[\psi](x) = 0 \ (x \notin \partial\Omega)$$

Integral equation to be solved

$$\frac{\partial}{\partial \nu} S[\psi]|_{-} = \left(-\frac{1}{2} I + \mathcal{K}^*_{\partial\Omega}\right)[\psi] = g \quad \text{(Fredholm equation)}$$

$$\Downarrow$$

From $\sigma_p(\mathcal{K}^*_{\partial\Omega}) = \sigma_p(\mathcal{K}_{\partial\Omega}) \subset (-\frac{1}{2}, \frac{1}{2}]$, we have ψ for $g \in L^2_0(\partial\Omega)$.

§1.2 Introduction and Motivation (Applications)

Why spectral geometry of NP operators ? (Heuristics)

(Metamaterial) $\nabla(\sigma\nabla)u + \omega^2 u = 0$, $\omega \approx 0$ (quasi-static)

$$\sigma = \begin{cases} 1 & \text{in } \mathbb{R}^n \backslash \Omega \\ \epsilon & \text{in } \Omega \end{cases} \qquad \begin{cases} u|_+ = u|_- & \text{(Continuity of potential)} \\ \frac{\partial u}{\partial \nu}|_+ = \epsilon \frac{\partial u}{\partial \nu}|_- & \text{(Continuity of flux)} \end{cases}$$

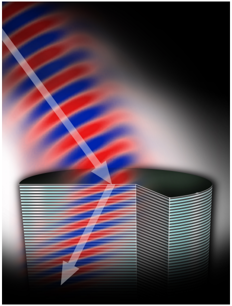
Representation of sol. $u(x) = h(x) + S[\psi](x)$, $h(x) \approx 0$.

$$\frac{\partial}{\partial \nu} S[\psi]|_+ = \epsilon \frac{\partial}{\partial \nu} S[\psi]|_-$$

By Jump formulas $\frac{\partial}{\partial \nu} S[\psi]|_\pm = (\pm\frac{1}{2} I + \mathcal{K}^*_{\partial\Omega})[\psi]$, we obtain

$$\left(\frac{\epsilon+1}{2(\epsilon-1)} I - \mathcal{K}^*_{\partial\Omega}\right)[\psi] = 0 \quad \text{on } \partial\Omega$$

$$-\frac{1}{2} < \frac{\epsilon+1}{2(\epsilon-1)} < \frac{1}{2} \Leftrightarrow \epsilon < 0 \,!?$$

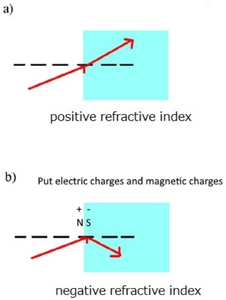


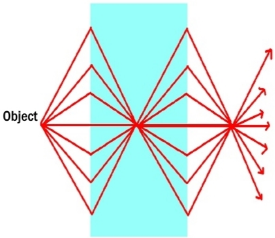
Keith Drake, NSF (2007)

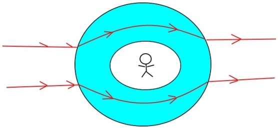
§1.2 Introduction and Motivation (Applications)

From negative permittivity ϵ and negative magnetic permeability μ,

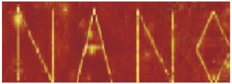
$$n = -\sqrt{\epsilon}\sqrt{\mu}$$ negative refractive index







Cloaking



super resolution using 20nm flat silver layer
Xiang et al., Nature Materials (2008)

§1.3 Introduction (Basic Properties)

- $\mathcal{K}_{\partial\Omega} : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is compact.
- $\sigma_p(\mathcal{K}_{\partial\Omega}) \subset (-\frac{1}{2}, \frac{1}{2}]$: real numbers
- The eigenvalue $\frac{1}{2}$ corresponds to constant eigenfunctions.
- If $n = 2$, the eigenvalues of $\mathcal{K}_{\partial\Omega}$ are symmetric with respect to the origin except for $\frac{1}{2}$.
- If $n = 2$ and $\partial\Omega$ is C^2 smooth, the kernel $\nu_y E(x, y) \in C(\partial\Omega \times \partial\Omega)$

(Example 1) ($n = 2$, Circle)

$$\partial\Omega = S^1 \ \Rightarrow \sigma_p(\mathcal{K}_{\partial\Omega}) = \{\tfrac{1}{2}, 0\}.$$

(Example 2) ($n = 2$, Ellipse)

$$\partial\Omega = \{(x, y) \mid \tfrac{x^2}{a^2} + \tfrac{y^2}{b^2} = 1 \ (a > b)\}$$

$$\Rightarrow \sigma_p(\mathcal{K}_{\partial\Omega}) = \{\tfrac{1}{2}, \pm\tfrac{1}{2}(\tfrac{a-b}{a+b})^j, \ j \in \mathbb{N}\}.$$

(Example 3) ($n = 3$, Sphere)

$$\partial\Omega = S^2 \ \Rightarrow \sigma_p(\mathcal{K}_{\partial\Omega}) = \{\tfrac{1}{2}, \tfrac{1}{2(2j+1)}, \ j \in \mathbb{N}\}.$$

§1.4 Introduction (Eigen- and Singular-values)

(Eigenvalues)

$$\sigma_p(\mathcal{K}_{\partial\Omega}) = \{\ \lambda_j \mid \frac{1}{2} = \lambda_0 > |\lambda_1| \geq |\lambda_2| \geq\ \cdots\}$$

where

$$\mathcal{K}_{\partial\Omega} e_{\lambda_j}(x) = \lambda_j e_{\lambda_j}(x).$$

(Singular values)

Recall that every compact operator K on L^2 space takes the canonical form

$$K\psi = \sum_{j=1}^{\infty} s_j(K)\langle \psi, v_j\rangle u_j$$

for some orthonormal basis $\{u_j\}$ and $\{v_j\}$, where $s_j(K)$ are singular values of K (i.e., the eigenvalues of $(K^*K)^{1/2}$), and $\langle\cdot,\cdot\rangle$ is the $L^2(\partial\Omega)$ inner product. We denote the ordered singular values of $\mathcal{K}_{\partial\Omega}$ by

$$\sigma_{sing}(\mathcal{K}_{\partial\Omega}) = \{\ s_j(\mathcal{K}_{\partial\Omega}) \mid s_1(\mathcal{K}_{\partial\Omega}) \geq s_2(\mathcal{K}_{\partial\Omega}) \geq\ \cdots\}.$$

§1.5 Introduction (Problems)

(i) What can we say about the geometry of $\partial\Omega$ given the eigen- or singular values?

(ii) What can we say about the eigenvalues, singular values, and eigenfunctions given the geometry?

(Spectral geometry of $-\triangle$)

	Spectral geometry of $-\triangle$ bounded region Ω in $\mathbf{R}^n$
Isoperimetric (Spectrum)	Faber-Krahn ineq. Szegő-Weinberger ineq.
Isoperimetric (Schatten norm)	The trace of Riesz potential is minimized in a ball
Eigenvalue asymptotics	Weyl's law $\lambda_j \approx \frac{4\pi^2 j^{2/n}}{(C_n\|\Omega\|)^{2/n}}$ as $j \to \infty$

§2 Summary of Results 1 (Spectral geometry) ($n = 2$)

	Spectral geometry of $-\triangle$ bounded region Ω in $\mathbf{R}^n$	$\mathbf{2}$-dimensional double layer potentials
Isoperimetric (Spectrum)	Faber-Krahn ineq. Szegő-Weinberger ineq.	$\sigma_p(\mathcal{K}_{\partial\Omega}) = \{1/2, 0\}$ $\sigma_{sing}(\mathcal{K}_{\partial\Omega}) = \{1/2, 0\}$
Isoperimetric (Schatten norm)	The trace of Riesz potential is minimized in a ball	tr$(\mathcal{K}^*_{\partial\Omega}\mathcal{K}_{\partial\Omega})= 1$
Eigenvalue asymptotics	Weyl's law $\lambda_j \approx \frac{4\pi^2 j^{2/n}}{(C_n\|\Omega\|)^{2/n}}$ as $j \to \infty$	Let Ω be a C^k region. For any $\alpha > -k + 3/2$ $\lambda_j = o(j^\alpha)$ as $j \to \infty$

(Q1) $\lambda_j \approx C j^{-k+3/2}$ as $j \to \infty$? If so, what is C?

(Q2) If $\partial\Omega$ is analytic, the exponential decay of eigenvalues ? (Yes)

(Q3) Isospectral domains ? (conformal map) (n=2 $\Rightarrow$ Shiffer)

(Q4) What can we say about asymptotics of eigenfunctions?

(Q5) Neumann-Poincaré operator on manifolds $\cdots$ etc.

§2.1 Comments on Results ($n = 2$)

(Decay properties)

Theorem (T. Suzuki and Y. M., Trans. AMS (2016))

Let $n = 2$ and $\partial\Omega$ be a C^k $(k \geqq 2)$ boundary. For any $\alpha > -k + 3/2$,

$$s_j(\mathcal{K}_{\partial\Omega}) = o(j^{\alpha}) \text{ and } \lambda_j = o(j^{\alpha}) \quad \text{as } j \to \infty,$$

where o indicates the small order.

It follows that if $\partial\Omega$ is a C^∞ boundary, then $\lambda_j = o(j^{-\infty})$.

Theorem (K. Ando, H. Kang and Y. M., Int. Eq. Appl. (2017))

Let $n = 2$ and $\partial\Omega$ be analytic. Let G_ϵ be a maximal Grauert tube of K. For any $\alpha < \epsilon$,

$$\lambda_j = o(e^{-\alpha j}) \quad \text{as } j \to \infty.$$

(Remark) These estimates are optimal (in some sense).

§2.1 Comments on Results ($n = 2$)

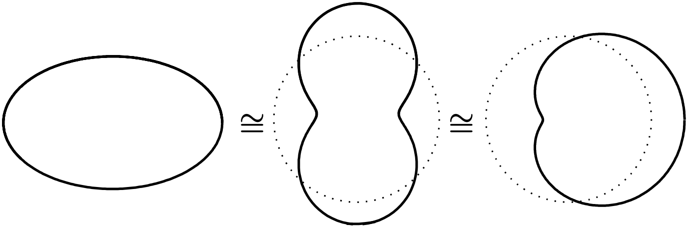
(Möbius Invariance of spectrum and Maximal Grauert radius)

(Example 2) ($n = 2$, Ellipse)

$$\partial\Omega = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)\}$$

$$\Rightarrow \sigma_p(\mathcal{K}_{\partial\Omega}) = \{\frac{1}{2}, \pm\frac{1}{2}(\frac{a-b}{a+b})^j, \ j \in \mathbb{N}\}.$$

Isospectral curves (Möbius invariant)



$$\sigma_p(\mathcal{K}_{\text{Ellipse}}) = \sigma_p(\mathcal{K}_{\text{Hippopedo}}) = \sigma_p(\mathcal{K}_{\text{Limaçon de Pascal}})$$

The maximal Grauert radius $\epsilon = \left|\log(\frac{a-b}{a+b})\right|.$

§3. Our Main Concern (Three dimensions)

(Example 3) ($n = 3$, Sphere)

$$\partial\Omega = S^2 \quad \Rightarrow \sigma_p(\mathcal{K}_{\partial\Omega}) = \{\tfrac{1}{2}, \tfrac{1}{2(2k+1)},\ k \in \mathbb{N}\}.$$

Counting multiplicities, the eigenvalues are denoted by

$$\underbrace{\frac{1}{2}}_{1}, \underbrace{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}}_{3}, \underbrace{\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}}_{5}, \cdots, \underbrace{\frac{1}{2(2k+1)}, \cdots, \frac{1}{2(2k+1)}}_{k},$$

So the $j = k^2$th eigenvalue satisfies

$$\lambda_j(\mathcal{K}_{S^2}) = \frac{1}{2(2k+1)} \sim \frac{1}{4} j^{-1/2} \quad \text{(Weyl's law in three dimensions).}$$

What happens for general surfaces?

§4 Main Theorem (Three dimensions)

(Notation)

$$H(x) : \text{Mean curvature of } \partial\Omega$$

$$\chi(\partial\Omega) : \text{Euler characteristics of } \partial\Omega$$

Under these notations, we have

Theorem (Weyl's law for $n = 3$ (Y. M., Arxiv: 1806.03657))

Let Ω be a $C^{k,\alpha}$ bounded domain in $\mathbb{R}^3$ with $k \geq 2$ and $\alpha > 0$. Then

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \sim \left\{\frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi}\right\}^{1/2} j^{-1/2} \quad \text{as } j \to \infty.$$

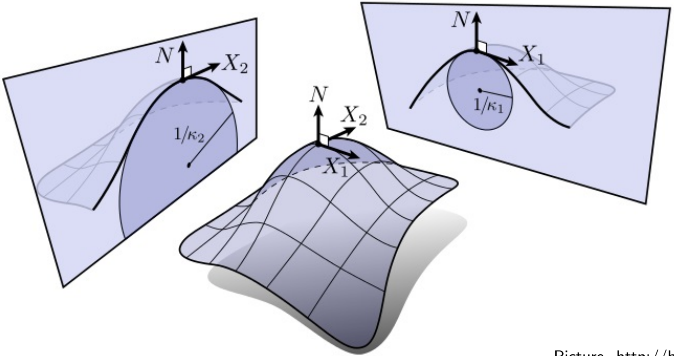
Here $W(\partial\Omega) := \int_{\partial\Omega} H^2(x)\, dS_x$ (Willmore energy).

Thus the principal part of decay rate is independent of the boundary smoothness.

§4.1. Mean curvature and Willmore energy

Mean curvature is the average of principal curvatures

$$\boldsymbol{H(x) = \frac{\kappa_1(x) + \kappa_2(x)}{2}}$$



Picture. http://brickisland.net

The Willmore energy $\boldsymbol{W(\partial\Omega) = \int_{\partial\Omega} H^2(x)\, dS_x}$

is invariant under Möbius transforms !! W. Blaschke (1929) etc.

§4.2. Willmore energy and applications

(Example 3) ($n = 3$, Sphere)

$$\partial\Omega = S^2 \quad \Leftrightarrow W(S^2) = 4\pi \text{ and } \chi(S^2) = 2.$$

Thus

$$|\lambda_j(\mathcal{K}_{S^2})| \sim \left\{\frac{3W(S^2) - 2\pi\chi(S^2)}{128\pi}\right\}^{1/2} j^{-1/2} = \frac{1}{4} j^{-1/2}$$

Conversely if we know the asymptotics of eigenvalues,

Corollary

Let $\Omega \subset \mathbb{R}^3$ be a bounded region of class $C^{2,\alpha}$.

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \sim \frac{1}{4} j^{-1/2} \Leftrightarrow \partial\Omega = S^2.$$

Especially

$$\sigma_p(\partial\Omega) = \sigma_p(S^2) \;\Leftrightarrow \partial\Omega = S^2.$$

We can hear the shape (sphere)!!

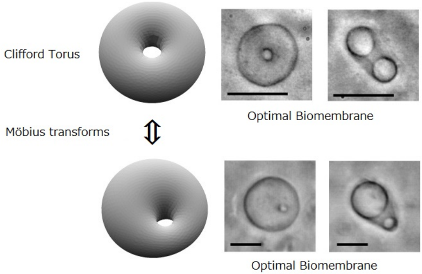
§4.2. Willmore energy and applications

(Example 4) ($n = 3$ with genus g) For higher genus cases, F. C. Marques and A. Neves (2014) proved the famous “Willmore conjecture”

$$W(\partial\Omega) \geq 2\pi^2.$$

The equality is achieved by the torus of revolution whose generating circle has radius 1 and center at distance $\sqrt{2}$ from the axis of revolution:

$$T^2_{\text{Clifford}} := \{((\sqrt{2} + \cos u)\cos v, (\sqrt{2} + \cos u)\sin v, \sin u) \in \mathbb{R}^3\}$$



M. Mutz and D. Bensimon (1992).

§4.2. Willmore energy and applications

For $g \geq 1$

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \sim \left\{\frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi}\right\}^{1/2} j^{-1/2} \geq \frac{\sqrt{3\pi}}{8} j^{-1/2}$$

The equality holds for the Clifford torus up to conformal transformations of $\mathbf{R}^3$.

Corollary

Let $\Omega \subset \mathbb{R}^3$ be a bounded region of class $C^{2,\alpha}$.

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \lessapprox \frac{\sqrt{3\pi}}{8} j^{-1/2} \Leftrightarrow \partial\Omega \text{ is simply connected.}$$

If the genus $g \geq 1$, then

$$\sigma_p(\partial\Omega) = \sigma_p(T^2_{\mathit{Clifford}}) \Rightarrow \partial\Omega \cong T^2_{\mathit{Clifford}}.$$

Here $\cong$ means the modulo Möbius transforms.

§4.2. Willmore energy and applications

(Example 5) (genus $g \geq 1$)
For a knot torus, Langevin and Rosenberg (1976) proved

$$W(\partial\Omega) \geq 8\pi.$$


https://bakingandmath.com

(Some other properties)

- If $\partial\Omega$ doesn't contain umblic points

$$W(\partial\Omega) > 16\pi.$$

- $\mathcal{W}(\partial\Omega, g) := \inf_{\substack{\partial\Omega:\ C^2\text{smooth} \\ g(\partial\Omega)=g}} W(\partial\Omega)$: The energy minimizer for fixed g.

$$\mathcal{W}(\partial\Omega, 0) = 4\pi < \mathcal{W}(\partial\Omega, 1) = 2\pi^2 < \mathcal{W}(\partial\Omega, 2) \cdots < 8\pi.$$

$$\lim_{g\to\infty} \mathcal{W}(\partial\Omega, g) = 8\pi.$$

§4.3 Applications (electro-static NP operators)

Khavinson-Putinar-Shapiro (2007) propose a question :
“ The disk is the only planar domain for which the NP operator has finite rank. It is not known whether there are such domains in higher dimensions ”

Corollary (Finite-rank problem)

Let Ω be a $C^{2,\alpha}$ region in $\mathbb{R}^n$ $(n = 2, 3)$. If $\mathcal{K}_{\partial\Omega}$ is finite rank, then

$$n = 2 \text{ and } \partial\Omega = S^1.$$

Thus the finite rank NP operator is rank one.

§4.3 Applications (Plasmonic eigenvalues)

(Plasmon equation)

$$\begin{aligned} \triangle u &= 0 && \text{in } \mathbf{R}^3 \backslash \partial\Omega \\ u_- &= u_+ && \text{on } \partial\Omega \\ \epsilon \partial_n u_- &= -\partial_n u_+ && \text{on } \partial\Omega. \end{aligned}$$

Here $u|_\Omega$ and $u|_{\mathbf{R}^3 \backslash \overline{\Omega}}$ are bounded Dirichlet energy functions and $\epsilon \in \mathbf{R}$ (often called "the plasmonic eigenvalue") satisfies

$$\epsilon_j = \frac{-\lambda_j(\mathcal{K}_{\partial\Omega}) - 1/2}{\lambda_j(\mathcal{K}_{\partial\Omega}) - 1/2} \sim 1 + 4\lambda_j(\mathcal{K}_{\partial\Omega}).$$

Thus

$$|\epsilon_j - 1| \sim \left\{ \frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{8\pi} \right\}^{1/2} j^{-1/2}.$$

§5.1 Strategy of Proof

(Weyl's law for singular values of $\mathcal{K}_{\partial\Omega}$)

(Step 1) Construct a pseudodifferential operator $Op(\sigma)$ on $\partial\Omega$, which is $\mathcal{K}_{\partial\Omega}$ modulo Hilbert-Schmidt class operator $\mathbf{H}$.

(Step 2) The symbol

$$\sigma(x,\xi) = \frac{L(x)\xi_2^2 - 2M(x)\xi_1\xi_2 + N(x)\xi_1^2}{2\det(g_{ij}(x))\{\sqrt{\sum_{j,k} g^{jk}(x)\xi_j\xi_k}\}^3}$$

is homogeneous order -1 with respect to ξ.

(Step 3) From Birman-Solomyak theorem, $s_j(Op(\sigma)) \sim C(\partial\Omega)j^{-1/2}$.

(Step 4) $s_j(\mathbf{H}) = o(j^{-1/2})$.

(Step 5) From Ky-Fan theorem and Gil's theorem (2007), $\lambda_j(\mathcal{K}_{\partial\Omega}) \sim s_j(\mathcal{K}_{\partial\Omega}) \sim C(\partial\Omega)j^{-1/2}$.

§5.2 (Step 3) Proof of Main Theorem

(Weyl's law for compact ψDO)
Let us introduce Weyl's law of singular values of the pseudo-differential operators with C^α smooth in x-variable. Birman and Solomyak showed the asymptotics under weak smoothness hypothesis both in the x- and ξ-variable. In our situation, Weyl's law is given by

Theorem (Birman-Solomyak (1977), Grubb (2014))

On a closed manifold M of dimension 2, let P be defined in local coordinates from symbols $\sigma(x,\xi)$ that are homogeneous in ξ of degree -1. Assume that the symbols restricted to $\xi \in S^{m-1} = \{|\xi| = 1\}$ are in $C(S^{m-1}, C^\epsilon)$ for some ϵ. Then

$$s_j(P) \sim C^{1/2} j^{-1/2} \quad j \to \infty.$$

Here

$$C = \frac{1}{8\pi^2} \int_{S^*M} |\sigma(x,\xi)|^2 dx\, d\xi \tag{1}$$

§5.3 (Step3) The constant $C(\partial\Omega)$

Let us take the isothermal charts.

$$\mathbf{I} = e^{2\sigma}(ds^2 + dt^2) \quad (\text{i.e. } E = G = e^{2\sigma},\ F = 0) \tag{2}$$

$$\begin{aligned}
C(\partial\Omega)^2 &= \frac{1}{8\pi^2}\int_{\partial\Omega}\int_{S^1}\left[\frac{L(x)\xi_2^2 - 2M(x)\xi_1\xi_2 + N(x)\xi_1^2}{4\det(g_{ij})\{\sqrt{\sum_{j,k} g^{jk}(x)\xi_j\xi_k}\}^3}\right]^2 d\xi dx \\
&= \frac{1}{8\pi^2}\int_{\partial\Omega}\int_{S^1}\left[\frac{L(x)\cos^2\theta - 2M(x)\cos\theta\sin\theta + N(x)\sin^2\theta}{4E^2(x)E^{-3/2}(x)}\right]^2 d\theta dx \\
&= \frac{1}{128\pi^2}\int_{\partial\Omega}\int_{S^1}\frac{(L(x)\cos^2\theta - 2M(x)\cos\theta\sin\theta + N(x)\sin^2\theta)^2}{E(x)}\, d\theta dx \\
&= \frac{1}{128\pi^2}\int_{\partial\Omega}\frac{(\frac{3\pi}{4}L^2(x) + \frac{3\pi}{4}N^2(x) + \pi M^2(x) + \frac{\pi}{2}L(x)N(x))}{E(x)}\, dx \\
&= \frac{1}{128\pi^2}\int_{\partial\Omega}\frac{(\frac{3\pi}{4}L^2(x) + \frac{3\pi}{4}N^2(x) + \pi(L(x)N(x) - E^2(x)K(x)) + \frac{\pi}{2}L(x)N(x))}{E(x)}\, dx \\
&= \frac{3}{512\pi}\int_{\partial\Omega}\Big[\left(\frac{L(x)+N(x)}{E(x)}\right)^2 - \frac{4}{3}K(x)\Big]E(x)\, dx \\
&= \frac{3}{512\pi}\int_{\partial\Omega}4H^2(x)\, dx - \frac{1}{64}\chi(\partial\Omega) \\
&= \frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi}.
\end{aligned}$$

§5.3 (Step5) Proof of Main Theorem

Definition (Schatten norms)

$$\|K\|_{S^p} = \mathrm{tr}\{(K^*K)^p\} \equiv \sum_j |s_j(K)|^{2p}$$

Lemma (Weyl's inequality)

$$\sum_{\lambda_j \in \sigma_p(K)} |\lambda_j|^{2p} \leq \sum_{s_j(K) \in \sigma_{sing}(K)} |s_j(K)|^{2p}$$

Lemma (Ky-Fan theorem)

If $\mathcal{K}_{\partial\Omega} = A + \mathrm{H},\ s_j(A) \sim Cj^{-1/2}, \mathrm{H}$ *is Hilbert-Schmidt op. Then*

$$s_j(\mathcal{K}^*_{\partial\Omega}) \sim Cj^{-1/2}.$$

§5.3 (Step5) Proof of Main Theorem

Theorem (M. I. Gil' (2007))

Let $\boldsymbol{K}$ *be in pth-Schatten for* $\boldsymbol{p > 2}$. *Then for* $\boldsymbol{2 < p \leq 4}$,

$$\left[\sum_{j=1}^{\infty} |\mathrm{Re}\boldsymbol{\lambda}_j(\boldsymbol{K})|^p\right]^{\frac{1}{p}} + (1+\sqrt{2})^{\frac{p-2}{2}} \left[\sum_{j=1}^{\infty} |\mathrm{Im}\boldsymbol{\lambda}_j(\boldsymbol{K})|^p\right]^{\frac{1}{p}}$$

$$\geq \left\|\frac{K+K}{2}\right\|_{S^p} - (1+\sqrt{2})^{\frac{p-2}{2}} \left\|\frac{K-K^*}{2i}\right\|_{S^p}$$

Theorem

Assume the following (1)-(3):

(1) $\boldsymbol{K}$ *is in pth-Schatten only for* $\boldsymbol{p > 2}$ *and* $\boldsymbol{K - K^*}$ *is Hilbert-Schmidt.*

(2) *Eigenvalues of* $\boldsymbol{K}$ *consist of real values.*

(3) $s_j(K) \sim Cj^{-1/2}$ *as* $j \to \infty$.

Then $\lambda_j(K) \sim s_j(K) \sim Cj^{-1/2}$ *as* $j \to \infty$.

§6 Perspective (Weyl's formulas)

(Weyl's law for *signed* eigenvalues of $\mathcal{K}_{\partial\Omega}$)

We define the *signed* Browder-Gårding density as

$$C_{\pm}(\partial\Omega) = \frac{1}{8\pi^2}\int_{S^*(\partial\Omega)} \sigma^{\pm}(x,\xi)^2 dx\, d\xi.$$

Here $\sigma^{\pm}$ denote the positive and negative part respectively. Then

Unsolved Problem (Weyl's law for *signed* eigenvalues)

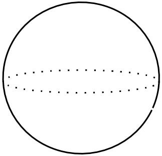
Let Ω be a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^3$. Then the spectrum of the NP operator on $\partial\Omega$ consists of eigenvalues $\lambda_j(\partial\Omega)$ which converge to 0, and

$$\lambda_j(\mathcal{K}_{\partial\Omega})^{\pm} \sim \pm C_{\pm}(\partial\Omega)^{1/2} j^{-1/2} \quad \textit{as } j \to \infty? \tag{3}$$

§6 Perspective (Weyl's formulas)

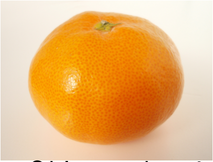
(Weyl's law for signed eigenvalues of $\mathcal{K}_{\partial\Omega}$)

(If Weyl's law for signed eigenvalues holds true!)
The above problem is strongly related with the Gaussian curvature of $\partial\Omega$. Indeed, if there exists a genus of $\partial\Omega$ then $C_\pm > 0$ from Gauss-Bonnet theorem. So we can find many negative eigenvalues. We also remark that for the case of a special oblate spheroid, Ahner (1994) finds the negative eigenvalue $\lambda = -(0.0598615\cdots)/2 < 0$. This is an example of negative eigenvalues for a positively curved boundary with genus 0. It seems that these negative eigenvalues are very sparse.



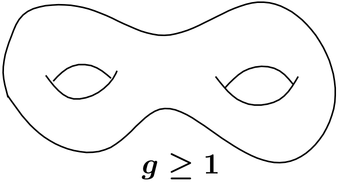
Sphere

Positive eigenvalues



Oblate spheroid

Sparse negative



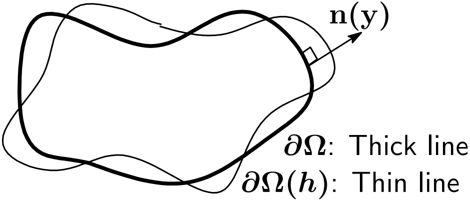
Many negative

§7 Hadamard's variational formula

Let Ω be a smooth bounded region in $\mathbf{R}^3$.
Fix Ω and let $a : \partial\Omega \to \mathbf{R}$ be a smooth function, and for real numbers h close to zero consider the bounded domain $\Omega(h)$ with smooth boundary

$$\partial\Omega(h) = \{\mathrm{x} + ha(\mathrm{x})\mathbf{n}(\mathrm{x}) \ : \ x \in \partial\Omega\}. \tag{4}$$

Then, $\Omega(0) = \Omega$ and $\Omega(h)$ is obtained by shifting the boundary in the normal direction $\mathbf{n}$ by ha. h is the order of magnitude of the shift and a is a "shape" function.



§7 Hadamard's variational formula

Theorem (The first Hadamard variation formula)

Let $\lambda \neq 0, 1/2$ *be an eigenvalue of the NP operator* $\mathcal{K}^*_{\partial\Omega}$ *with eigenspace* E*. Then there are* $h_0 > 0$ *and real smooth functions* $h \mapsto \lambda^{(i)}(h)$*,* $h \mapsto e^{(i)}(h)$ *defined for* $|h| < h_0$*,* $i = 1, \dots, \dim E$*, such that for each* h *the numbers* $\lambda^{(i)}(h)$*,* $i = 1, \dots, \dim E$ *are eigenvalues of* $\mathcal{K}^*_{\partial\Omega(h)}$ *with eigenfunctions* $e^{(i)}(h)$ *and* $e^{(1)}(0), \dots, e^{\dim E}(0)$ *are a basis of* E*.*

For a fixed smooth branch $\lambda(h)$*,* $e(h)$ *of eigenvalues and eigenfunctions with* $\lambda(0) = \lambda$ *and* $\|\nabla u(h)\|_{L^2(\Omega(h))} = \|\nabla \mathcal{S}_{\partial\Omega(h)}[e(h)]\|_{L^2(\Omega(h))} = 1$ *for each* h*, we have*

$$\dot{\lambda} := \frac{d}{dh}\lambda(0) = \int_{\partial\Omega} a\Big[\Big(\lambda - \frac{1}{2}\Big)|\nabla_\partial u_-|^2 + \Big(\lambda + \frac{1}{2}\Big)(\partial_n u_-)^2\Big] dS. \qquad (5)$$

Here $u = \mathcal{S}_{\partial\Omega}[e(0)]$ *and the index* $-$ *indicates the limits (to* $\partial\Omega$*) from* Ω*.* ∇_∂ *and* ∂_n *denote the decomposition of the standard (Euclidean) gradient* ∇*, namely,* $\nabla u = \nabla_\partial u + (\partial_n u)\mathrm{n}$ *on* $\partial\Omega$*.*

§7 Hadamard's variational formula (Application)

Theorem (Jointwork with K. Ando, H. Kang and E. Ushikoshi)

Let $\partial\Omega = S^2$ *be a standard sphere and* $\lambda_{k,l}(0) = \frac{1}{2(2k+1)}$ $(l = -k, \ldots, k)$ *be the degenerated eigenvalues of* $\mathcal{K}^*_{\partial\Omega}$*. Under smooth deformation of the boundary, the set of eigenvalues satisfies*

$$\sum_{l=-k}^{k} \dot{\lambda}_{k,l} := \sum_{l=-k}^{k} \frac{d}{dh}\lambda_{k,l}(0) = 0 \quad \textit{for any } k \in \mathbb{N}_{\geq 0}.$$

Thus the the sum of $2k+1$ eigenvalues is $1/2 + O(h^2)$. Here it is worth mentioning that the sum is identically $1/2$ for general ellipsoids (Equilibrium of eigenvalues).

Can one prove or disprove the $1/2$ conjecture and $1/6$ conjecture?

Remark. If $1/2$ conjecture is true, it follows $1/6$ conjecture is also true.

§8 Perspective (electro-static NP operators)

Conjecture ($1/2$ conjecture (The partial sum of eigenvalues is $1/2$))

Let $\partial\Omega$ be a simply connected surface. For any $n \in \mathbf{N}_{\geq 0}$, there exist $2n+1$ eigenvalues satisfying

$$\sum_{k=1}^{2n+1} \lambda_{j(k)}(\partial\Omega) = \frac{1}{2}.$$

More generally, even for multiply connected surfaces

$$\sum_{k=1}^{2n+1} \lambda_{j(k)}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega} K(x)\, dS = \frac{1}{2}(1-g).$$

Here $K(x)$ is the Gaussian curvature and g denotes the genus.

(The validity of this conjecture.) When C^{∞} closed surfaces are replaced by ellipsoids, this conjecture holds true (E. Martensen, (1999)).

§8 Perspective (electro-static NP operators)

Conjecture ($1/6$ conjecture (Isoperimetric property))

Let $n = 3$ *and* $\overline{\lambda} \equiv \max \sigma_p(\mathcal{K}_{\partial\Omega}) \setminus \{\frac{1}{2}\}$. *We have*

$$\inf_{\partial\Omega} \overline{\lambda} = \frac{1}{6},$$

where the infimum is taken over all C^∞ *simply connected closed surfaces. The infimum is achieved if and only if* $\partial\Omega = S^2$.

Conjecture (Isoperimetric property)

Let $n = 3$. *For* $p > 1$,

$$\inf_{\partial\Omega} \sum_j |\lambda_j(\mathcal{K}_{\partial\Omega})|^{2p} = \left(\frac{1}{2^{2p}} - \frac{1}{2^{4p-1}}\right) \zeta(2p-1),$$

where the infimum is taken over all C^∞ *simply connected closed surfaces and* $\zeta(s)$ *is the Riemann zeta function. The infimum is achieved if and only if* $\partial\Omega = S^2$.

(The validity of these conjectures.) When C^∞ closed surfaces are replaced by ellipsoids, these conjectures hold true.

References of this talk

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Thank you for your attention!