Fredholm conditions for pseudodifferential operators

– layer potentials –

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Main result: Informal statement

Theorem (Main result)

On a “nice” *manifold, a* “nice” *pseudo-differential operator is Fredholm if, and only if,*

1. *it is elliptic and*
2. *all its* **“limit operators”** *are invertible.*

(Carvalho-N.-Qiao, Lauter-Monthubert-N., earlier: Georgescu, HVZ, Kondratiev, Melrose, Schulze, Schrohe, ...)

Rest of the talk: we “define” the terms in quotation marks :-).

The presentation is (somewhat) geared towards layer potentials. Also: **polyhedral** ⇒ “nice.”

Summary

Collaborators: B. Ammann, C. Carvalho, B. Monthubert, M. Putinar*, Yu Qiao*

* = here in the audience.

Part I:

Motivation and earlier results

Ellipticity and the classical Fredholm criteria

Let $P =$ order m (classical, pseudo)differential op. w. bounded* coeff on a smooth man. M. (* $P : H^s(M) \to H^{s-m}(M)$ is bdd.)

Theorem (classical)

Assume that $M =$ is ***compact****. We have*

"P is Fredholm $\Leftrightarrow$ P is elliptic."

(No limit operators.)

("Elliptic" ⇔ princ. symb. invertible.)

NOT true on singular or non-compact spaces! Substitutes?

Essential spectrum

Let $X, Y =$ Banach spaces. A cont lin $P : X \to Y$ is **Fredholm** if

$$\ker(P) := \{Pu = 0\} \text{ and } \operatorname{coker}(P) := Y/PX$$

are **finite dimensional.**

Important question: determine the **essential spectrum.**

$$\sigma_{\text{ess}}(H) := \{\lambda \in \mathbb{C} \,|\, H - \lambda : D(H) \to L^2(M) \text{ is } \textbf{not} \text{ Fredh. } \}$$

Clue: classical N-body problem: **HVZ Theorem.** Also, the "*b*-calculus," the "edge-calculus."

Degenerate and singular PDEs

Δ in **Generalized spherical coordinates** (“cylindrical ends case” or the “*b*-calculus”):

$$\Delta = r^{-2}\big((r\partial_r)^2 + (n-2)r\partial_r + \Delta_{S^{n-1}}\big)\,.$$

In **cylindrical coordinates** (“asympt. hyperbolic” or the “edge calculus”):

$$\Delta = r^{-2}\big((r\partial_r)^2 + \partial_\theta^2 + (r\partial_z)^2\big)\,.$$

Closely related to the layer potentials on the corresponding geometries.

Generated by suitable **vector fields**: $\mathcal{V} = span\{r\partial_r, \partial_\theta, r\partial_z\}$.
These are the **main examples** in what follows.

Kernel spaces

An idea of treating layer potentials (and pseudodifferential operators in general) on a singular space $\overline{M}$ is to consider suitable **kernel spaces**.

Denote by M the **smooth** part of $\overline{M}$.

(Classical) pseudodifferential operators on M are given by distributions on $M \times M$ that are **conormal** to the diagonal M.

The kernels of pseudodifferential operators on $\overline{M}$ will **not** extend to distributions on $\overline{M} \times \overline{M}$ conormal w.r.t. the diagonal.

They often do extend, however, to **a different completion** $\mathcal{G}$ of $M \times M$ ("bigger" than $\overline{M} \times \overline{M}$). This completion is the **kernel space:** nice algebraic and geometric properties.

Main result: SECOND informal statement

Theorem (Main result)

Consider a "nice" manifold M and a "nice" order m pseudo-differential operator P:
If $m \leq 0$, then the **essential spectrum** *of P on $L^2(M)$ is*

$$\sigma_{ess}(P) = Im(\sigma_0(P)) \cup \cup_\alpha \sigma(P_\alpha)\,.$$

If $m > 0$ and P is elliptic, then $P : H^m(M) \to L^2(M)$ has

$$\sigma_{ess}(P) = \cup_\alpha \sigma(P_\alpha)\,.$$

Here the P_α are the "**limit operators**" of P.

Immediate consequence of the first formulation.

Layer potentials

If Ω is a bounded domain with **conical points**, then the layer potential operators K and rS are "nice". (Same for Δ.)

Their limit operators are the corresponding operators K and S associated to the tangent cones at the conical points.

There are as many "limit" operators as conical points.

The original theory was **not** developed with layer potentials in mind. The application to layer potentials is just an after thought.

That's why, in the presentation, differential operators come first.

More motivation

My (pers) init motivation: **well posedness on polyhedral domains** using layer potentials in weighted Sobolev spaces

$$\mathcal{K}^m_a(\Omega) := \{ u \mid r^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \leq m \},$$

where r is the distance to the skeleton of codimension 2 (edges in 3D, vertices in 2D).

Theorem (Bacuta-Mazzucato-V.N.-Zikatanov)

Let Ω be a (curvilinear) polyhedral domain (nD). Then there exists $\eta_\Omega > 0$ such that we have an isomorphism

$$\Delta : \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to \mathcal{K}^{m-1}_{a-1},\ m \in \mathbb{Z}_+,\ |a| < \eta_\Omega .$$

2D Kondratiev '67, $\eta_\Omega = \frac{\pi}{\alpha_{MAX}}$. Mazya-Rossmann: correct spaces and statement in 3D.

Part II:

Fredholm operators on manifolds with nice ends

Lie algebras of vector fields

Assume that we are given:

1. A **compact** manifold **with corners** $\overline{M}$, $M = \overline{M} \smallsetminus \partial\overline{M}$, ($\partial\overline{M}$ =the boundary of $\overline{M}$).
2. A subspace

$$\mathcal{V} \subset \mathcal{V}_b := \{X \in \mathcal{C}^\infty(\overline{M}; T\overline{M}) \mid \text{ tangent to all faces of } \overline{M}\}$$

The vector fields are regarded as first order differential operators without constant term.

A manifold with corners is locally of the form $[0, 1)^n$. Same as smooth manifolds (without boundary or corners).

Differential and pseudo-differential operators

Algebra of diff operators generated by $\mathcal{C}^\infty(\overline{M})$ and $\mathcal{V}$:

$$\mathrm{Diff}(\mathcal{V})\,.$$

There exists an associated **pseudo-differential calculus**

$$\Psi(\mathcal{V}) \text{ “ = ” } \mathrm{Diff}(\mathcal{V})[\,\Delta^{-1}\,]\,,$$

with the usual **symbolic properties** and

$$\Psi(\mathcal{V}) \cap \mathrm{Diff}(M) = \mathrm{Diff}(\mathcal{V})\,.$$

There is a suitable **completion** $\mathcal{G}$ of $M \times M$ such that the kernels of the operators in $\Psi(\mathcal{V})$ **are the conormal distributions** on $\mathcal{G}$ (conormal w.r.t $\overline{M}$).

Orbits and isotropy

Let $\{Z_\alpha\}$ be the **orbits** of $\mathcal{V}$ on $\partial\overline{M} := \overline{M} \smallsetminus M$. Fix $x \in Z_\alpha$ and

$$m_x \ := \ \{\, f \in \mathcal{C}^\infty(M) \mid f(x) = 0 \,\}\,.$$

Clearly, $m_x\mathcal{V} \subset \{\, X \in \mathcal{V} \mid X(x) = 0 \,\} \ =: \ \mathcal{V}_x$.

The **isotropy** of $\mathcal{V}$ at x is then the quotient

$$I_x \ := \ \mathcal{V}_x / m_x\mathcal{V}\,.$$

It depends only on α, $x \in Z_\alpha$ (up to isomorphism, in general). Let G_α the simply connected Lie group with Lie algebra I_x.

Fredholm conditions on “nice” manifolds

Assume $(M, \mathcal{V})$ is “nice” and $D \in \Psi^m(\mathcal{V})$. To M and D we can associate

1. spaces Z_α, $\alpha \in I$ (orbits, independent of D);
2. groups G_α, $\alpha \in I$ (independent of D); and
3. G_α-invariant differential operator D_α on $Z_\alpha \times G_\alpha$ s.t.

Theorem (Ammann-Lauter-V.N., Lauter-Monthubert-V.N., Carvalho-V.N.-Qiao)

$D : H^s(M) \to H^{s-m}(M)$ is Fredholm $\Leftrightarrow$ D is elliptic and all D_α, $\alpha \in I$, are invertible.

Example: polygons

We do not have a characterization of “nice” pairs $(M, \mathcal{V})$, but anything that comes from **polyhedral domains** is nice.

Let us see what we obtain in the case of a **polygon** Ω.

Let $\Sigma'(\Omega)$ be the **disjoin union** of its edges (sides). Take *one* such side I and assume $I = [0, \infty)$. Then

- $\mathcal{V} = \mathcal{V}_b(I) = \{\, tf(t)\partial_t \mid f \in \mathcal{C}^\infty(I) \,\}$.
- $\overline{M} = [0, \infty)$, $M = (0, \infty)$, and
- $\mathcal{G} \simeq [0, \infty) \times (0, \infty)$ with $(x, t) \in (0, \infty)^2$ identified with $(x, tx) \in (0, \infty)^2$. (Recall that $\mathcal{G}$ is a completion of $M \times M$.)

Layer potentials

If $I = [0, 1]$, we take

$$\mathcal{V} = \mathcal{V}_b(I) = \{t(1-t)f(t)\partial_t \mid f \in \mathcal{C}^\infty(I)\}.$$

The resulting pseudodifferential calculus is the "*b*"-calculus: Lewis-Parenti, Melrose, Schulze, ...).

The same is true for cones (below, Yu, Putinar-Perfekt, Bonnetier, Hyeonbae Kang, Putinar-Nistor-Yu, Nistor-Yu, ...).

Three examples of “nice” manifolds

Assume $\overline{M} = [0, \epsilon) \times \partial\overline{M} \ni (r, x)$.

1. **Cylindrical ends (“*b*”-calculus):**

$$\mathcal{V} = \mathcal{V}_b = \mathcal{C}^\infty(\overline{M}) r\partial_r + \sum \mathcal{C}^\infty(\overline{M})\partial_x \,.$$

2. **Asymptotically hyperbolic (edges):**

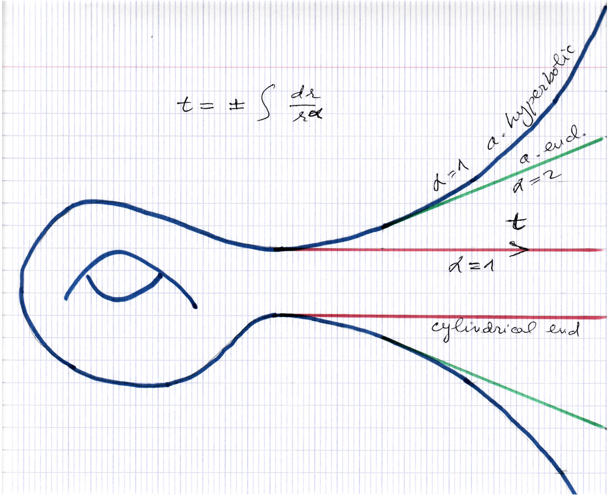
$$\mathcal{V} = \mathcal{C}^\infty(\overline{M}) r\partial_r + \sum \mathcal{C}^\infty(\overline{M}) r\partial_x \,.$$

3. **Asymptotically euclidean (conic):**

$$\mathcal{V} = \mathcal{C}^\infty(\overline{M}) r^2\partial_r + \sum \mathcal{C}^\infty(\overline{M}) r\partial_x \,.$$

Examples

$dt := \frac{dr}{r^\alpha}$; a. cyl. & hyp.: $\alpha = 1$; a. euclidean $\alpha = 2$.



Fredholm conditions for the three examples of "nice" manifolds

1. **Cylindrical ends ($g = \frac{(dr)^2}{r^2} + h$):**
   $I = \{*\},\ \ Z_\alpha = \partial\overline{M},\ \ G_\alpha = \mathbb{R}$.
2. **Asymptotically hyperbolic ($g = \frac{(dr)^2+h}{r^2}$):**
   $I = \partial\overline{M},\ \ Z_\alpha = \{\alpha\},\ \ G_\alpha = T_\alpha\partial\overline{M}\rtimes\mathbb{R}$, **non-comm.**
3. **Asymptotically conical (euclidean, $g = \frac{(dr)^2}{r^4} + \frac{h}{r^2}$):**
   $I = \partial\overline{M},\ \ Z_\alpha = \{\alpha\},\ \ G_\alpha = T_\alpha\partial\overline{M}\times\mathbb{R}$, commutative.

Cones

For a straigth cone,

$$\mathcal{C} := \{rx' \mid r \geq 0, x' \in \omega \subset S^{n-1}\} \subset \mathbb{R}^n ,$$

the relevant algebra on the boundary comes from

- $\overline{M} = \partial\omega \times [0, \infty]$;
- $\mathcal{V} = \mathcal{V}_b$ (v.f. tangent to $\{0, \infty\} \times \partial\omega$);
- $\mathcal{G} = \partial\omega \times \partial\omega \times [0, \infty] \times (0, \infty)$.

We again get the *b*-calculus and

$$K, \; rS \; \in \; \Psi^{-1}(\overline{M}, \mathcal{V}).$$

Summary and conclusion

- For “nice” manifolds, the Fredholm condition is controlled by ellipticity and by the invertibility of the limit operators.
- Many examples (a. hyperbolic, the N-body problem, ...). Closed under blow-ups.
- Challenge: to prove that the layer potential operators on polyhedral domains belong to the corresponding calculi (and hence are “nice”).

  True for domains with conical points and (tentatively) for three dimensional polyhedral domains.

Thank you!