The spectrum of the Neumann–Poincaré operator on domains with corners and conical points

Karl-Mikael Perfekt

joint w/ Johan Helsing and Mihai Putinar

University of Reading

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Layer potentials for the Laplacian

- $\Gamma \subset \mathbb{R}^3$ a connected Lipschitz surface with surface measure $d\sigma$, or a Lipschitz curve $\Gamma \subset \mathbb{R}^2$, enclosing a bounded open domain $\mathrm{int}(\Gamma)$.
- Layer potential operator

$$K^\Gamma f(\boldsymbol{r}) = \frac{1}{2\pi}\,\mathrm{p.v.}\int_\Gamma K^\Gamma(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,d\sigma(\boldsymbol{r}'), \quad \boldsymbol{r} \in \Gamma,$$

  based on the normal derivative of the Newtonian kernel

$$K^\Gamma(\boldsymbol{r},\boldsymbol{r}') = \frac{\langle \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{\nu}_{\boldsymbol{r}}\rangle}{|\boldsymbol{r}'-\boldsymbol{r}|^3}\ (3\mathrm{D}), \quad K^\Gamma(\boldsymbol{r},\boldsymbol{r}') = 2\frac{\langle \boldsymbol{r}-\boldsymbol{r}', \boldsymbol{\nu}_{\boldsymbol{r}}\rangle}{|\boldsymbol{r}'-\boldsymbol{r}|^2}\ (2\mathrm{D}),$$

  where $\boldsymbol{\nu}_{\boldsymbol{r}}$ denotes the outward unit normal of Γ.
- Adjoint $(K^\Gamma)^*$ w.r.t. $L^2(\Gamma)$-pairing is called the *Neumann–Poincaré operator*.

Many spectra

- If Γ is non-smooth, the spectrum of $K^\Gamma : X \to X$ depends on X, the space for the boundary data.
- If $1 < p \leq 2$ and Γ Lipschitz, then $K^\Gamma \pm I : L^p(\Gamma) \to L^p(\Gamma)$ is always Fredholm. Verchota '84, after Calderón and Coifman, McIntosh, Meyer '82 proved boundedness.
- If $\Gamma \subset \mathbb{R}^2$ is a curvilinear polygon, then there always exist $p_0 > 2$, depending on the opening angles, such that $K^\Gamma \pm I : L^{p_0}(\Gamma) \to L^{p_0}(\Gamma)$ is not Fredholm.

Many spectra

- Reason: If $\Gamma \subset \mathbb{R}^2$ is an infinite wedge, the model domain for a corner in 2D. Then $K^\Gamma : L^p(\Gamma) \to L^p(\Gamma)$ isometrically identifies with a block matrix of Mellin convolution operators with different kernels for different p.
- Fabes, Jodeit, Lewis '87: Similar results for $\Gamma \subset \mathbb{R}^3$ when Γ is an infinite straight cone (rotationally symmetric conical point).
- Polyhedrons are surprisingly complicated, little is known in the non-elliptic case.

Single layer potentials and the energy space

- Single layer potential

$$S^\Gamma f(\boldsymbol{r}) = \int_\Gamma S^\Gamma(\boldsymbol{r},\boldsymbol{r}') f(\boldsymbol{r}')\, d\sigma(\boldsymbol{r}'), \quad \boldsymbol{r} \in \mathbb{R}^3,$$

where

$$S^\Gamma(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{2\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\ \text{(3D)}, \quad S^\Gamma(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{\pi}\log\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\ \text{(2D)}.$$

- Energy space $\mathscr{E}$: distributions f such that

$$\|f\|_{\mathscr{E}}^2 = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla S^\Gamma f|^2\, dV = \langle S^\Gamma f, f\rangle_{L^2(\Gamma)} < \infty.$$

$\mathscr{E}$ non-isometrically identifies with $H^{-1/2}(\Gamma)$, the Sobolev space of index $-1/2$ on Γ.

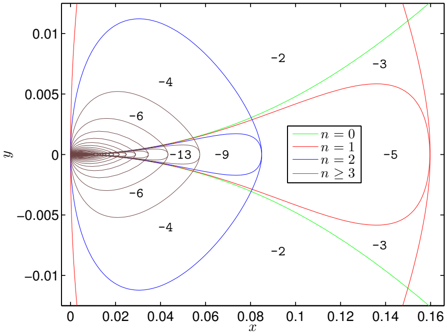
Single layer potentials and the energy space

$$\langle f, g \rangle_{\mathscr{E}} = \langle S^\Gamma f, g \rangle_{L^2(\Gamma)} = \frac{1}{2} \int_{\mathbb{R}^3} \langle \nabla S^\Gamma f, \nabla S^\Gamma g \rangle_{\mathbb{R}^3} \, dV.$$

- Plemelj formula:
$$S^\Gamma K^\Gamma = (K^\Gamma)^* S^\Gamma$$
- $K^\Gamma : \mathscr{E} \to \mathscr{E}$ is self-adjoint, and hence has a real spectrum (the spectrum on $L^2(\Gamma)$ is only real when Γ is smooth).

Real vs non-real spectrum

$\Gamma \subset \mathbb{R}^3$ a bounded domain with conical point of opening angle 2α.



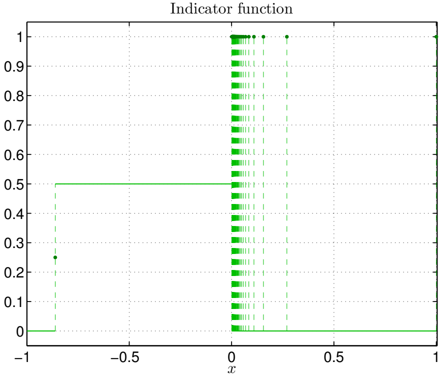


Figure: (a): Spectrum on $L^2(\Gamma)$ $(2\alpha = 5\pi/18)$. (b): Spectrum on $\mathscr{E}$ for mode $n = 0$ $(2\alpha = 31\pi/18)$.

Spectrum of K^Γ and the transmission problem

- Transmission problem

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla U|^2 \, dV < \infty, \\ \Delta U(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \mathbb{R}^3 \setminus \Gamma, \\ \mathrm{Tr}_{\mathrm{int}} U(\boldsymbol{r}) = \mathrm{Tr}_{\mathrm{ext}} U(\boldsymbol{r}), \quad \boldsymbol{r} \in \Gamma, \\ \partial_{\boldsymbol{\nu}}^{\mathrm{ext}} U(\boldsymbol{r}) = \epsilon_r \partial_{\boldsymbol{\nu}}^{\mathrm{int}} U(\boldsymbol{r}) - g(\boldsymbol{r}), \quad \boldsymbol{r} \in \Gamma. \end{cases}$$

  where $\epsilon_r \neq 1$ and $g \in \mathscr{E}$.

- Ansatz $U = S^\Gamma f$: equivalent to find $f \in \mathscr{E}$ such that

$$(K^\Gamma - z) f = \frac{g}{1 - \epsilon_r}, \quad z = -\frac{1 + \epsilon_r}{1 - \epsilon_r}.$$

- Special case: $g(\boldsymbol{r}) = \boldsymbol{e} \cdot \boldsymbol{\nu}_{\boldsymbol{r}}$, $\boldsymbol{e} \in \mathbb{R}^3$. Involved in computing the polarizability tensor of $\mathrm{int}(\Gamma)$ – blow-up corresponds to permittivities ϵ for which surface plasmon resonances can be excited.

2D: Curvilinear polygons

- Curvilinear polygon $\Gamma \subset \mathbb{R}^2$ with a finite number of corners $(a_j)_{j=1}^N$ with (interior) angles $(\alpha_j)_{j=1}^N$.
- Zaremba (1904), Carleman (1916), and Radon (1919): Essential spectral radius of $K^\Gamma : C(\Gamma) \to C(\Gamma)$ is given by

$$\left|\sigma_{\text{ess}}(K, C(\Gamma))\right| = \max_{1 \le j \le N} \left|1 - \frac{\alpha_j}{\pi}\right|.$$

2D: Curvilinear polygons

Shelepov '69, Lewis '91, I. Mitrea '02: Essential spectrum of $K^\Gamma : L^p(\Gamma) \to L^p(\Gamma)$, $1 < p < \infty$, consists of the union of "bowtie domains", one for each corner of Γ. The Fredholm index $\mathrm{index}(K^\Gamma - z)$ is the winding number of z w.r.t. the essential spectrum. All eigenvalues must be real.

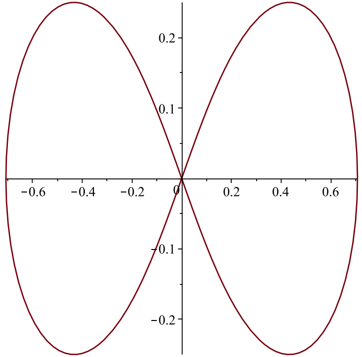


Figure: $p = 2$, one corner of angle $\alpha = \pi/2$. $\mathrm{index}(K^\Gamma - z) = -1$ inside bowties.

2D: The Ahlfors-Beurling transform

Let

$$L^2_a(\mathrm{int}(\Gamma)) = \left\{ f \in \mathrm{Hol}(\mathrm{int}(\Gamma)) : \int_{\mathrm{int}(\Gamma)} |f(z)|^2 \, dA(z) < \infty \right\}, \quad dA(z) = dx\,dy.$$

The anti-linear Ahlfors–Beurling transform $T_\Gamma : L^2_a(\mathrm{int}(\Gamma)) \to L^2_a(\mathrm{int}(\Gamma))$ is given by

$$T_\Gamma f(z) = \frac{1}{\pi} \, \mathrm{p.v.} \int_{\mathrm{int}(\Gamma)} \frac{\overline{f(\zeta)}}{(\zeta - z)^2} \, dA(\zeta), \quad f \in L^2_a(\mathrm{int}(\Gamma)), \, z \in \mathrm{int}(\Gamma).$$

Theorem (Bergman, Schiffer 1951)

If $x \in \mathbb{R}$ and $x \neq 1$, then x is an eigenvalue of K^Γ if and only if it is an eigenvalue of T_Γ.

The Ahlfors-Beurling transform

Theorem (Costabel 2007 and Khavinson, Putinar, Shapiro 2007)

If Γ is Lipschitz, then $K^{\Gamma}: \mathscr{E}_0 \to \mathscr{E}_0$ is similar to T_{Γ} (as an $\mathbb{R}$-linear operator). In particular

$$\sigma(K^{\Gamma}, \mathscr{E}) = \sigma_{\mathbb{R}}(T_{\Gamma}) \cup \{1\}.$$

Theorem (Ahlfors 1952)

The Ahlfors inequality: If Γ is a quasi-circle, then

$$\|T_{\Gamma}\| \leq q_{\Gamma},$$

where q_{Γ} is the quasi-conformal reflection coefficient of Γ. In particular, if Γ is not a circle or a line, then $\|T_{\Gamma}\| < 1$.

2D: Essential spectrum

Theorem (P., Putinar, 2017)

Let $K^\Gamma : \mathscr{E} \to \mathscr{E}$ be the adjoint of the Neumann–Poincaré operator, for a C^2-smooth curvilinear polygon $\Gamma \subset \mathbb{C}$ with angles $\alpha_1, \ldots, \alpha_N$. Then

$$\sigma(K^\Gamma, \mathscr{E}) = \left\{ x \in \mathbb{R} : |x| \leq \max_{1 \leq j \leq N} \left| 1 - \frac{\alpha_j}{\pi} \right| \right\} \cup \{\lambda_k\}.$$

*where $\{\lambda_k\}$ is a sequence of **real** eigenvalues with no limit point outside the interval. The spectrum is symmetric around $x = 0$, except for the eigenvalue $\lambda = 1$.*

2D: Proof strategy

- Step 1 (**complex analysis**): Compute the spectrum of T_{W_α} for the wedge
$$W_\alpha = \{z \in \mathbb{C} : |\arg z| = \alpha/2\}.$$
Based on Fourier analysis of $L^2_a(\mathrm{int}(W_\alpha))$ via group of unitaries
$$U_t f(z) = e^t f(e^t z).$$
Due to homogeneity of kernel:
$$U_t T_{W_\alpha} = T_{W_\alpha} U_t.$$
- Step 2 (**trick**): Localize the operator K^{W_α} to each corner (the wedge has two corners!).

2D: Proof strategy

- Step 3 (**complex analysis**): Perturbation argument to deal with curvilinear corners, not just corners arising from two straight lines. Uses conformal mapping and results on their regularity.

- Step 4 (**real analysis, operator theory**): Localize the operator K^{Γ} to the corners of any curvilinear polygon.

2D: refinements

Theorem (H. Kang, M. Lim, S. Yu, 2017)

For the wedge W_α, explicit spectral resolution (i.e. diagonalization) of $K^{W_\alpha}: \mathcal{E} \to \mathcal{E}$. Real variable techniques, by understanding explicitly the scalar product

$$\langle f, g \rangle_{\mathcal{E}} = \langle S^{W_\alpha} f, g \rangle_{L^2(W_\alpha)}.$$

Theorem (E. Bonnetier, H. Zhang, arXiv 2017)

A different approach to the theorem, leading to an understanding how the singular part of the solution ρ to $(K^\Gamma - z)\rho = g$ behaves when $z \notin \sigma_{\text{ess}}(K^\Gamma, \mathcal{E})$.

3D: Rotationally symmetric conical points

Let Γ be a closed surface of revolution with a conical point of opening angle 2α, obtained by revolving a C^5-curve γ.

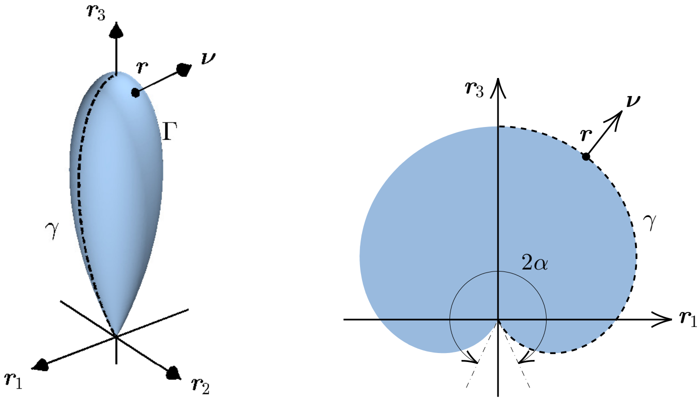


Figure: (a): An axially symmetric surface Γ with a conical point of opening angle $2\alpha = 5\pi/18$. (b): A cross-section of $\text{int}(\Gamma)$ with opening angle $2\alpha = 31\pi/18$.

3D: Modal operators

- Parametrization:

$$\boldsymbol{r}(t,\theta) = (\gamma_1(t)\cos\theta, \gamma_1(t)\sin\theta, \gamma_2(t)), \qquad \theta \in [0,2\pi],\ 0 \le t \le 1.$$

- Kernel on K^Γ rotationally invariant,

$$K^\Gamma(t,\theta,t',\theta') = K^\Gamma(t,\theta-\theta',t',0).$$

- Fourier mode kernels:

$$K_n^\gamma(t,t') = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} K^\Gamma(t,\theta,t',0)\,d\theta, \qquad 0 < t,t' \le 1.$$

defining operators

$$K_n^\gamma f_n(t) = \int_0^1 K_n^\gamma(t,t') f_n(t')\gamma_1(t')|\gamma'(t')|\,dt', \quad 0 \le t \le 1.$$

3D: Modal operators

- For $K^\Gamma\colon L^2(\Gamma) \to L^2(\Gamma)$,

$$K^\Gamma \simeq_{\text{ue}} \bigoplus_{n=-\infty}^{\infty} K_n^\gamma,$$

where $K_n^\gamma : L^2(\gamma_1(t)|\gamma'(t)|\,dt) \to L^2(\gamma_1(t)|\gamma'(t)|\,dt)$.

- For $K^\Gamma\colon \mathscr{E} \to \mathscr{E}$,

$$K^\Gamma \simeq_{\text{ue}} \bigoplus_{n=-\infty}^{\infty} K_n^\gamma,$$

where $K_n^\gamma : \mathscr{E}_n \to \mathscr{E}_n$. Here

$$\|f_n\|_{\mathscr{E}_n}^2 = \langle S_n^\gamma f_n, g_n\rangle_{L^2(\gamma_1(t)|\gamma'(t)|\,dt)},$$

where S_n^γ is the modal version of S^Γ.

3D: Spectrum on $L^2(\Gamma)$

For $n \in \mathbb{Z}$, denote by Π_n the closed curve

$$\Pi_n = \left\{ \frac{P^n_{i\xi}(\cos\alpha)\frac{dP^n_{i\xi}}{dx}(-\cos\alpha) - P^n_{i\xi}(-\cos\alpha)\frac{dP^n_{i\xi}}{dx}(\cos\alpha)}{P^n_{i\xi}(-\cos\alpha)\frac{dP^n_{i\xi}}{dx}(\cos\alpha) + P^n_{i\xi}(\cos\alpha)\frac{dP^n_{i\xi}}{dx}(-\cos\alpha)} : -\infty \le \xi \le \infty \right\},$$

with orientation given by the ξ-variable. Here $P^n_\lambda(x)$ denotes an associated Legendre function of the first kind.

3D: Spectrum on $L^2(\Gamma)$

Theorem (Helsing, P., in press)

The operator $K^\Gamma : L^2(\Gamma, d\sigma) \to L^2(\Gamma, d\sigma)$ *has essential spectrum*

$$\sigma_{\text{ess}}(K^\Gamma, L^2) = \bigcup_{n=-\infty}^{\infty} \Pi_n.$$

If $z \notin \sigma_{\text{ess}}(K^\Gamma, L^2)$, *then* $K^\Gamma - z$ *has Fredholm index*

$$\text{index}(K^\Gamma - z) = \sum_{n=-\infty}^{\infty} W(z, \Pi_n) = W(z, \Pi_0) + 2\sum_{n=1}^{\infty} W(z, \Pi_n)$$

where $W(z, \Pi_n) \le 0$ *denotes the winding number of* z *with respect to* Π_n *and the right-hand side is always a finite sum.*
All eigenvalues z *must be* ***real****.*

3D: Spectrum on $L^2(\Gamma)$

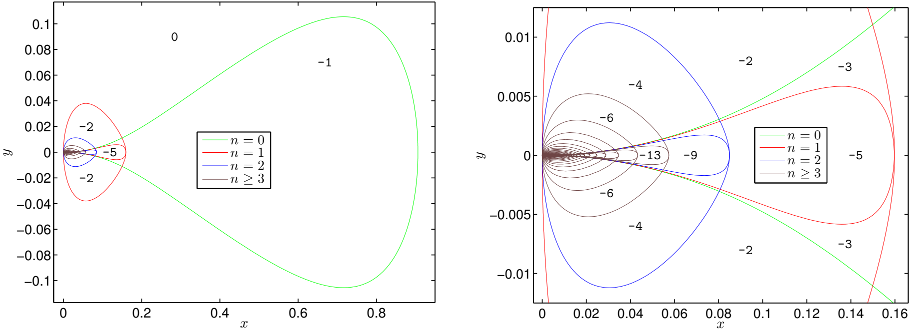


Figure: The essential spectrum of K^Γ for a surface Γ with a conical point of opening angle $2\alpha = 5\pi/18$. index$(K^\Gamma - z)$ is marked in the regions of non-essential spectrum. Zoom to the right.

3D: Spectrum on $\mathscr{E}$

Theorem (Helsing, P., in press)

For $n \in \mathbb{Z}$, denote by $\Sigma_n \subset \mathbb{R}$ the closed interval

$$\Sigma_n = \left\{ \frac{P^n_{i\xi-1/2}(\beta)\frac{dP^n_{i\xi-1/2}}{dx}(-\beta) - P^n_{i\xi-1/2}(-\beta)\frac{dP^n_{i\xi-1/2}}{dx}(\beta)}{P^n_{i\xi-1/2}(-\beta)\frac{dP^n_{i\xi-1/2}}{dx}(\beta) + P^n_{i\xi-1/2}(\beta)\frac{dP^n_{i\xi-1/2}}{dx}(-\beta)} : -\infty \le \xi \le \infty \right\},$$

where $\beta = \cos\alpha$. Then the self-adjoint operator $K^\Gamma : \mathscr{E} \to \mathscr{E}$, where $\mathscr{E}$ is the energy space of Γ, has essential spectrum

$$\sigma_{\text{ess}}(K^\Gamma, \mathscr{E}) = \bigcup_{n=-\infty}^{\infty} \Sigma_n.$$

Hence, the spectrum of K^Γ consists of this interval and a sequence of real eigenvalues $\{\lambda_k\}$ with no limit point outside of it,

$$\sigma(K^\Gamma, \mathscr{E}) = \{\lambda_k\} \cup \sigma_{\text{ess}}(K^\Gamma, \mathscr{E}).$$

3D: Spectrum on $\mathscr{E}$

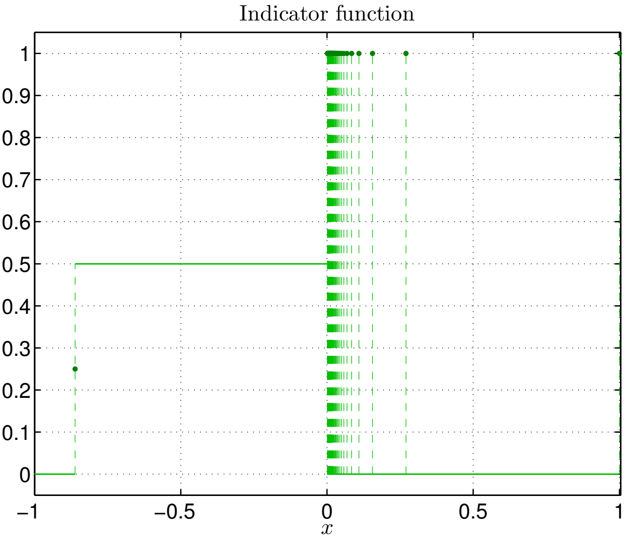


Figure: Spectrum on $\mathscr{E}$ for mode $n = 0$ $(2\alpha = 31\pi/18)$.

Step 1: Model case

- Model case: Γ_α a straight infinite circular cone. Homogeneity:

$$K_n^{\gamma_\alpha}(\lambda t, \lambda t') = \frac{1}{\lambda^2} K_n^{\gamma_\alpha}(t, t'), \qquad \lambda > 0.$$

- Hence

$$K_n^{\gamma_\alpha} f(t) = \sin(\alpha) \int_0^\infty K_n^{\gamma_\alpha}\left(\frac{t}{t'}, 1\right) f(t') \frac{dt'}{t'},$$

a convolution against the group $\mathbb{R}^+$ with Haar measure $\frac{dt}{t}$. Let

$$F_n(\zeta) = \sin(\alpha) \mathscr{M} K_n^{\gamma_\alpha}(\zeta) = \sin(\alpha) \int_0^\infty t^\zeta K_n^{\gamma_\alpha}(t, 1) \frac{dt}{t}.$$

It turns out that

$$\Pi_n = \{F_n(1 + i\xi) : \xi \in \mathbb{R}\}, \quad \Sigma_n = \{F_n(3/2 + i\xi) : \xi \in \mathbb{R}\}$$

Step 2: Localization of model case

- Localize K^{Γ_α} to each conical point. The cone Γ_α has **two conical points**, equal but running in opposite directions.

Step 3: Perturbation

- Show that difference between K^{Γ_α} and K^{Γ} is compact, where Γ has a curvilinear corner.
- For $L^2(\Gamma)$: write difference of kernels as

$$\sum (\text{smooth kernels with decay at corner}) \times (\text{Riesz kernels}).$$

- The Riesz kernels do **not** define bounded operators on $\mathscr{E}$. How does one see that $K^{\Gamma}: \mathscr{E} \to \mathscr{E}$ is bounded? Use algebraic identities + real interpolation.

Step 4: Localization in general

- On $L^2(\Gamma)$, use the symbolic calculus for pseudodifferential Mellin type operators. Developed by Elschner, Lewis, Parenti. Index formula comes from their work.
- On $\mathscr{E}$ we use Weyl sequences. Very weak method that yields no information on the character of the spectrum.