Carleman's legacy in the spectral analysis of the Neumann-Poincaré operator

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Torsten Carleman

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Aim

To investigate the *first boundary problem* (i.e. Dirichlet problem) in the line of C. Neumann solution via double layer potentials, as refined by Poincaré (Acta Mathematica vol. 20 (1895)).

Continuation of studies of Korn, Zaremba (Journal de Math. 1904) and Steklov, and use of novel ideas in the theory of integral equations streaming from Fredholm's method (1900 on). Also close connections with works of Plemelj (1911).

In the case of domains with corners Fredholm's theory (infinite determinants) does not work and it has to be adapted.

Part I

Notation: T is a simply connected domain with boundary C and corners at $P_1, \dots, P_n$ with respective inner angles $\alpha_1, \dots, \alpha_n$. By arc-length parametrization the corners correspond to $s_1, \dots, s_n$.

Single layer potentials

$$V(p) = \int_C f(t) \log \frac{1}{r_{pt}} dt,$$

with $f(s)$ continuous for $s \neq s_i$ and

$$|s - s_i|^{\kappa} |f(s)| < k, \quad 0 < \kappa < 1.$$

Green's formulas

For $f(s)$ as above, the total inner energy is finite

$$\int_D |\nabla(f)|^2 dxdy < \infty$$

and the same for the exterior domain D' if

$$\int_c f(s)ds = 0.$$

Double layer potential

$$W(p) = \int_C \frac{\cos(n_t r_{tp})}{r_{tp}} \mu(t) dt,$$

where the charge $\mu(t)$ is continuous at $t \neq s_i, \ 1 \leq i \leq n$.
If $\mu(t)$ has a discontinuity at s_i, then so does W...

The boundary problem

$$W_i(s) - W_e(s) - \lambda(W_i(s) + W_e(s)) = 2\pi f(s)$$

is equivalent to the integral equation

$$\mu(s) - \lambda \int_0^\ell \frac{\cos(n_t r_{tp})}{r_{tp}} \mu(t) dt = f(s), \quad s \neq s_i.$$

Closer look at the NP kernel

$$K(s,t) = \frac{\cos(n_t r_{tp})}{r_{tp}}$$

is continuous for $t \neq s_i$ and has singularities along the lines $s = s_i$.

For s, t close to s_i one finds

$$K(s,t) = \frac{\sin\alpha_i}{\pi} \frac{|s-s_i|}{|s-s_i|^2 + |t-s_i|^2 - 2|s-s_i||t-s_i|\cos\alpha_i} + R(s,t).$$

Global decomposition

$$K(s,t) = G(s,t) + H(s,t),$$

where

$$G(s,t) = \sum_{i=1}^{n} \frac{|s-s_i|}{|s-s_i|^2 + |t-s_i|^2 - 2|s-s_i||t-s_i|\cos\alpha_i}$$

and the kernel $H(s,t)$ satisfies condition (E):

$H(s,t)$ is uniformly bounded and has singularities of the first kind along finitely many vertical and horizontal lines.

Uniform estimates for the unperturbed kernel

Let

$$g_i(s,t) = \frac{|s-s_i|}{|s-s_i|^2 + |t-s_i|^2 - 2|s-s_i||t-s_i|\cos\alpha_i}.$$

Then

$$\int_0^\ell g_i(s,t)||t-s_i|^{-\kappa_i}dt \leq \frac{\sin\kappa_i|\pi-\alpha_i|}{\sin\kappa_i\alpha_i}|s-s_i|^{-\kappa_i}.$$

and similarly for the adjoint. Hence equation

$$\phi(s) - \lambda\int_0^\ell G(s,t)\phi(t)dt = f(s),$$

has a solution (in the same function space), analytically depending on λ, for

$$|\lambda| < R' = \min_i \frac{\sin\kappa_i\alpha_i}{\sin\kappa_i|\pi-\alpha_i|}.$$

The unperturbed resolvent

Let

$$R = \min_i \frac{\alpha_i}{|\pi - \alpha_i|},$$

corresponding to the case all $\kappa_i \to 0$.
The integral operator with kernel G admits a resolvent $G(\lambda)$ with kernel $G(s, t/\lambda)$, given by a convergent series in the disk $|\lambda| < R$.
In modern language:

$$(I - \lambda G)^{-1} = I + \lambda G(\lambda).$$

with precise control of the continuity/regularity of $G(s, t/\lambda)$.

The perturbed resolvent

Starting from

$$K = G + H$$

one considers

$$\Omega(\lambda) = (I - \lambda G)^{-1} H$$

and

$$F(\lambda) = (I - \lambda G)^{-1} f,$$

as intermediate steps in solving

$$(I - \lambda K)\phi = f.$$

New integral equation

For $|\lambda| < R$ one has to solve

$$(I - \lambda\Omega(\lambda))\phi = F(\lambda).$$

Here Carleman proposes to use Fredholm's determinant method.

Fredholm method

In modern terms: let T be a trace class integral operator

$$(Tu)(x) = \int T(x,y)u(y)dy.$$

Then

$$D(\lambda) = \det(I - \lambda T)$$

exists, it is an entire function of exponential type zero, and

$$D(\lambda) = \prod_j (1 - \lambda_j \lambda), \quad \lambda \in \mathbb{C}.$$

Moreover:

$$(I - \lambda T)^{-1} = I + \lambda \frac{D_\lambda(T)}{D(\lambda)}$$

where $D_\lambda(T)$ is an operator valued entire function whose Taylor coefficients can be computed recurrently.

The determinants

Specifically, let

$$T\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \end{pmatrix} = \det T(x_i, y_j).$$

Then

$$D(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \int \dots \int T\begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_1 & x_2 & y_3 & \dots & x_n \end{pmatrix} dx_1 dx_2 \dots dx_n$$

and the kernel of the resolvent

$$D_\lambda(T)(x,y) =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int \int \dots \int T\begin{pmatrix} x & x_1 & x_2 & \dots & x_n \\ y & x_1 & x_2 & \dots & x_n \end{pmatrix} dx_1 dx_2 \dots dx_n.$$

The convergence problem

The residual kernel $H(s,t)$ may not be trace class!

Glitch resolved five years later by Carleman (1921), following the theory of "regularized determinants", founded by Plemelj (1904) and Poincaré (1909) .

This is the source of the theory of Carleman kernels (Akhiezer, 1947) and modern adaptation to quantum theory (Simon, 1977).

Conclusion- general theory

The integral equation with NP kernel

$$(I - \lambda K)\phi = f$$

has discrete spectrum, of finite multiplicity, inside the disk $|\lambda| < R = \min_i \frac{\alpha_i}{|\pi - \alpha_i|}$.

Both spaces of continuous functions with discontinuities at $s = s_i$, as well as "the case of Hilbert, Schmidt, Weyl" square integrable functions are discussed.

The same conclusion for the "Poincaré fundamental functions = energy norm" was reached before by Zaremba (1904).

Generalization of Hilbert's work

$K(s,t)$ is a *symmetric* kernel, continuous on the square $[0,\ell]\times[0,\ell]$ minus finitely many monotonic curves, and

$$\sup_s \int |K(s,t)|dt = \frac{1}{R} < \infty.$$

Then, in modern language, the associated integral equation admits only point spectrum of finite multiplicity in the interval $|\lambda| < R$.

Proof by an early variant of the min-max principle. (a la Weyl, in the words of Carleman).

Part II. The circular lune

The domain bounded by two circular arcs C_1, C_2 of different radii, intersecting at P_1, P_2. Using bipolar coordinates:

$$|pP_1| = \rho_1, \ |pP_2| = \rho_2, \ |P_1P_2| = a,$$

and

$$\mathrm{angle}(P_1pP_2) = \theta,$$

we work with $u = \log \rho_1 - \log \rho_2$, in the (inner) domain:

$$(u, \theta), \quad u \in \mathbb{R}, \quad \theta_1 < \theta < \theta_2,$$

Fourier analysis

The functions

$$(Ae^{q\theta} + Be^{-q\theta})\cos(qu), (Ce^{q\theta} + De^{-q\theta})\sin(qu)$$

are harmonic.

Ansatz. The boundary values of the double layer potentials can be written as

$$W_i(u,\theta) = \int_0^\infty (A_i(q)e^{q\theta} + B_i(q)e^{-q\theta})\cos(qu)dq +$$

$$(C_i(q)e^{q\theta} + D_i(q)e^{-q\theta})\sin(qu)dq,$$

and similarly for $W_e(u,\theta)$.

The solution

The NP equation

$$\mu(s) - \lambda \int_0^{\ell} \frac{\cos(n_t r_{tp})}{r_{tp}} \mu(t) dt = f(s)$$

admits the solution

$$\mu_1(u) = f_1(u) + \frac{\lambda}{\pi}[H_{F_1}(u, \theta/\lambda) - H_{F_2}(u, \theta/-\lambda)] + k,$$

where

$$2F_1(u) = f_1(u) + f_2(u), \quad 2F_2(u) = f_1(u) - f_2(u),$$

and similarly on C_2...

Key transform

$$H_f(u, \gamma/\lambda) = \int_{\mathbb{R}} \int_0^{\infty} \frac{\sinh \gamma q}{\sinh \pi q - \lambda \sinh \theta q} \cos q(u-\kappa) f(\kappa) dq d\kappa.$$

Analysis of singularity

When $q \in [0, \infty)$ the spectral parameter

$$\lambda = \frac{\sinh \pi q}{\sinh \theta q} \in [\frac{\pi}{\theta}, \infty).$$

Notation

$$\Gamma = \mathbb{C} \setminus [\frac{\pi}{\theta}, \infty).$$

$$\theta = \pi - (\theta_2 - \theta_1).$$

Detailed analysis leads to the analyticity of $H_f(u, \gamma/\lambda)$ as a function of $\lambda \in S$, where

$$S = \mathbb{C} \setminus \{\lambda; |\lambda\theta| > \pi\}.$$

The integral equation

$$\mu(s) - \lambda \int_0^b [G(s,t) - G(2b-s,t)]\mu(t)dt = f(s),$$

where $G(s,t)$ has "corner singularities" at (s_1, s_1) and (s_2, s_2).

Reduction to a single corner:

$$f(s) = -f(2b-s).$$

...

Study of

$$\mu(s) = f(s) + \lambda \int_0^b N(s, t/\lambda) f(t) dt$$

with a precise resolvent N

Complex spectrum

Pass to complex domain

$$O_1 = \{q : \Re q \geq 0,\ 0 \leq \Im q \leq 1.\},$$

and conformal mapping

$$O_2 = \{\lambda = \frac{\sinh \pi q}{\sinh \theta q},\ q \in O_1\}.$$

At the boundary point

$$q(\lambda) = \sqrt{\lambda - \frac{\pi}{\theta}}\mathrm{holom}(\lambda).$$

For real $\lambda \in (\frac{\pi}{\theta}, \infty)$ the homogeneous equation

$$\mu(s) - \lambda \int_0^b N(s,t)\mu(t)dt = 0$$

admits a solution

$$U(s,\lambda) = \sin q(\lambda)u,$$

defined on both branches $\theta = \theta_1, \theta = \theta_2$. For non-real λ there are no integrable solutions

The synthesis

On a domain with one corner one osculates with a circular lune, of the same angle, and one splits the NP kernel $H(s,t)$ into

$$H(s,t) = N(s,t) + P(s,t),$$

with N the kernel along the circles.
Perturbation theory for

$$I - \lambda H = (I - \lambda P)[I - \lambda(I - \lambda P)^{-1}N]$$

and return to Fredholm determinants.

The zeros $\lambda_1, \lambda_2, \ldots,$ of $\det(I - \lambda P)$ are real, discrete.

... after an impressive amount of details...

Main result

For a domain with one corner, the NP equation

$$\phi - \lambda K \phi = 0$$

has a continuous solution iff $\lambda \in \{\lambda_1, \lambda_2, \dots\}$. These eigenvalues have finite multiplicity.
Moreover, for real values of the spectral parameter $|\lambda\theta| > \pi$ the NP equation admits solutions which, close to the corner $x = x_1$ are

$$\phi(x) = a(\lambda)\cos[q(\lambda)\log\frac{1}{|x-x_1|}] + b(\lambda)\sin[q(\lambda)\log\frac{1}{|x-x_1|}] + \phi_0(x,\lambda),$$

where $a(\lambda), b(\lambda), \phi_0(x,\lambda)$ are analytic in λ and the function ϕ_0 is continuous at $x = x_1$.

And similarly for the adjoint equation.

Section 11: Symmetrisable kernels

First the "smooth" setting.

$$K(x,y), \quad x,y \in [a,b]$$

is a Hilbert-Schmidt kernel so that

$$N(x,y) = \int_a^b S(x,t)K(t,y)dt$$

is symmetric, where $S(x,y)$ is also a symmetric kernel.

A *complete* and accurate discussion of the existence of eigenvalues of K (all real) and completeness of the eigenfunctions, in the spirit of Hilbert's recent articles.

Plus the exact evaluation of the genus of the associated regularized Fredholm determinant.

The eigenfunctions

Specifically, let (u_j) be an orthonormal basis of the "energy space"

$$\int\int S(x,y)u_i(x)u_j(y)dxdy = \delta_{ij}.$$

Then

$$\psi_j(x) = \int S(x,y)u_j(y)dy, \quad j \geq 0,$$

are the eigenfunctions of K:

$$\psi_j(x) - \lambda_j \int K(x,y)\psi_j(y)dy = 0.$$

Spectral decomposition

$$\int_a^b N(x,y)h(y)dy = \sum_0^\infty \frac{h_j}{\lambda_j}\psi_j(x),$$

with uniform convergence, for h square integrable (provided the kernel N is bounded), where

$$h_p = \int \psi_p(x)h(x)dx.$$

Moreover

$$\overset{(2)}{\det}(I-\lambda K) = e^{a\lambda}\prod_j (1-\frac{\lambda}{\lambda_j})e^{\frac{\lambda}{\lambda_j}},$$

a result due to Lalesco (thesis) and Privalov (1915).

With direct application to the NP kernel (with full details in dim 2 and 3).

Complemented by a long digression on the differentiability properties of (generalized) eigenfunctions.

A domain with one corner

Recall $\theta = \pi - \alpha$.

Let $\Phi(x, \lambda)$ be the "generalized" solution of

$$\phi(x) - \lambda \int K(x, y)\phi(y)dy = 0,$$

found for $|\lambda\theta| > \pi$.
Notation:

$$m(q) = \frac{\sinh \pi q}{\sinh \theta q}$$

and

$$\zeta(q) = \frac{\pi}{q} \frac{\cosh \pi q + \cosh \theta q}{\sinh \pi q}.$$

$$\Phi^*(x, \lambda) = \sqrt{\frac{\zeta(q)}{\pi m'(q)}} \Phi(x, \lambda).$$

The symmetric kernel

$$M(x,y) = \int_0^\ell K(x,z) \log \frac{R_0}{r_{zy}} dz.$$

Assume

$$\int_0^\ell \int_0^\ell \log \frac{R_0}{r_{zy}} |f(x)||f(y)| dxdy < \infty.$$

Let

$$\int_0^\ell \int_0^\ell \log \frac{R_0}{r_{zy}} \psi_j(x)\psi_k(y) dxdy = \delta_{jk}.$$

and

$$\phi_j(x) = \int_0^\ell \log \frac{R_0}{r_{zy}} \psi_j(y) dy,$$

$$f_i = \int_0^\ell \phi_i(x) f(x) dx, \quad f_\lambda = \int_0^\ell \Phi^*(x,\lambda) f(x) dx.$$

Second main result

$$\int_0^{\ell} M(x,y)f(y)dy = \sum_{j=0}^{\infty} \frac{f_i \phi_i(x)}{\lambda_i} + \int_{|\lambda\theta|>\pi} \frac{f_\lambda \Phi^*(x,\lambda)}{\lambda} d\lambda$$

with absolute convergence, and similarly

$$\int_0^{\ell} K(x,y)f(y)dy = \sum_{j=0}^{\infty} \frac{f_i \psi_i(x)}{\lambda_i} + \int_{|\lambda\theta|>\pi} \frac{f_\lambda \Psi^*(x,\lambda)}{\lambda} d\lambda$$

with a similar weight function Ψ^* ...

The solution of the non-homogeneous NP problem can be written as:

$$\psi(x) = f(x) + \lambda[\sum_{j=0}^{\infty} \frac{f_i \psi_i(x)}{\lambda_i - \lambda} + \int_{|\mu\theta|>\pi} \frac{f_\mu \Psi^*(x,\mu)}{\mu - \lambda} d\mu].$$

Sources

https://gdz.sub.uni-goettingen.de/

http://gallica.bnf.fr/

https://archive.org/