Double Layer Potentials on Polygons and Pseudodifferential Operators on Lie Groupoids

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Motivation Example

Let $\Omega \subset \mathbb{R}^n$ be a *bounded*, *open* set with boundary $\partial\Omega := \overline{\Omega} \setminus \Omega$. Let us consider on Ω the "simplest" boundary value problems, the Poisson problem

$$\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = g. \end{cases}$$

A well-known, classical result is the following "regularity theorem":

Theorem

If $\partial\Omega$ is smooth, then $\tilde{\Delta}(u) = (\Delta u, u|_{\partial\Omega})$ defines an isomorphism

$$\tilde{\Delta} : H^{m+2}(\Omega) \to H^m(\Omega) \oplus H^{m+\frac{3}{2}}(\partial\Omega),$$

for $m > -\frac{3}{2}$.

Note: it follows that if f, g, and $\partial\Omega$ are smooth, then u is also smooth (including the boundary).

Motivation Example

The above theorem is *NOT* true if $\partial\Omega$ is not smooth.

Let $\Omega = (0,1)^2$ and $g = 0$ on the boundary, and assume that u is smooth. Then we have

$$\partial_x^2 u(0,0) \ = 0 = \ \partial_y^2 u(0,0).$$

Therefore,

$$f(0,0) = \Delta u(0,0) = 0.$$

Remark

No solution on $(0,1)^2$ *is smooth if* $g = 0$ *and* $f(0,0) \neq 0$.

Conclusion: The nonsmoothness of the domain causes the problem.

C^*-algebras

Let H be a (separable) Hilbert space, $\mathcal{L}(H)$ the collection of all bounded linear operators $H \to H$. $\mathcal{L}(H)$ is equipped with an involution $*$, where the *adjoint* T^* of $T \in \mathcal{L}(H)$ is defined by

$$\langle T^* x, y \rangle = \langle x, Ty \rangle.$$

Moreover, $\mathcal{L}(H)$ is equipped with a norm

$$||T|| = \sup\{||T(x)|| : ||x|| \leqslant 1\},$$

which satisfies $||TT^*|| = ||T||^2$.

Definition

A C^*-algebra of operators on H is a norm-closed subalgebra A of $\mathcal{L}(H)$ such that $T \in A \Rightarrow T^* \in A$.

Example

Let X be a locally compact space. The algebra $C_0(X)$ of continuous functions vanishing at infinity is a C^*-algebra.

Review of Layer Potential Method

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and consider the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi, & \text{on } \partial\Omega \end{cases}$$

where Δ denotes Laplace operator.

The fundamental solution of Laplace's equation is given by

$$N(x) = \begin{cases} c_n |x|^{2-n} & n \geqslant 3 \\ c_2 \log |x| & n = 2, \end{cases}$$

where c_n is a constant depending only on n.

Review of Layer Potential Method

Denote $N(x,y) = N(x-y)$.

Let ψ be a function on $\partial\Omega$. (ψ could be in $L^2(\partial\Omega)$, $C(\partial\Omega)$, $\cdots$.) The double layer potential (with moment ψ) is defined by

$$(\mathcal{D}\psi)(x) \quad = \quad \int_{\partial\Omega} k(x,y)\psi(y)d\sigma(y), \qquad (x \in \mathbb{R}^n \backslash \partial\Omega)$$

where $k(x,y) = \partial_{\nu_y} N(x,y) = -\dfrac{c_n(x-y)\cdot\nu(y)}{|x-y|^n}$, and $\nu(y)$ is the unit outer normal vector to $\partial\Omega$ at y.

Fact: the function $\mathcal{D}\psi$ is harmonic.

Review of Layer Potential Method

Given a function u on $\mathbb{R}^n \backslash \partial\Omega$, for $x \in \partial\Omega$, let $u_+(x)$ and $u_-(x)$ denote the limits of $u(y)$ as $y \to x$ (nontanentially) from $y \in \Omega$ and $y \in \mathbb{R}^n \backslash \overline{\Omega}$, respectively. The classical results on double layer potentials give

Proposition

For (a.e.) $x \in \partial\Omega$, we have

$$(\mathcal{D}\psi)_+(x) = \psi(x) + K\psi, \quad (\mathcal{D}\psi)_-(x) = -\psi(x) + K\psi,$$

where

$$K\psi(x) = \int_{\partial\Omega} k(x,y)\psi(y)d\sigma(y).$$

The operator K is called the double layer potential operator.

Review of Layer Potential Method

We can try to solve the Dirichlet problem in terms of double layer potential

$$u(x) = (\mathcal{D}\psi)(x), \qquad x \in \Omega,$$

and relate ψ to φ. Letting $x \to z \in \partial\Omega$ yields

$$\varphi(z) = u(z) = (I + K)\psi(z), \qquad z \in \partial\Omega.$$

Hence, the (interior) Dirichlet problem is reduced to

$$\varphi = (I + K)\psi.$$

Remark

The (exterior) Dirichlet problem is reduced to

$$\varphi = (-I + K)\psi.$$

Review of Layer Potential Method

- If Ω has smooth boundary, then K is a pseudodifferential operator of order -1 on $\partial\Omega$, i.e., $K \in \Psi^{-1}(\partial\Omega)$. (Taylor's book.)
- If the boundary of Ω is C^2, then K is compact on $L^2(\partial\Omega)$. (Folland's book.)
- If the boundary of Ω is C^1, then K is compact on $L^p(\partial\Omega)$. (Fabes, Jodeit, and Riviere. Acta Math. 1978.)
- If Ω is a SKT domain, then K is compact. (Hofmann, Mitrea, and Taylor. 2010)
- If Ω is a Lipschitz domain, in general, K is not compact. However,

  $$I + K : L^2(\partial\Omega) \to L^2(\partial\Omega),$$

  is invertible. (Greg Verchota, 1984)

Question:

If a domain has singularities (polygon, polyhedra, curvilinear domains, ...), can we construct a (natural) pseudodifferential operator algebras containing K in an appropriate sense? (in other words, $K \in \Psi^{-1}(\text{something})$?)

Double Layer Potentials on Plane Sectors

Let $\Omega \subset \mathbb{R}^2$ be a plane sector with angle θ. So the boundary $\partial\Omega$ consists of two rays. The double layer potential operator K is of the form $\begin{pmatrix} 0 & K_{12} \\ K_{21} & 0 \end{pmatrix}$, where $K_{12} = K_{21} = \widetilde{K}$, and

$$(\widetilde{K}f)(t) \quad = \quad \int_0^\infty k(t/s)f(s)\frac{ds}{s},$$

and

$$k(t) = \frac{1}{\pi}\frac{t\sin\theta}{t^2 + 1 - 2t\cos\theta}.$$

Clearly, the operator $\widetilde{K}$ is a Mellin convolution operator on $\mathbb{R}^+$. Let M_f denote the multiplication operator by f. It is easy to see that $M_{r^a}\widetilde{K}M_{r^{-a}}$ has convolution kernel

$$k_a(r) = \frac{1}{\pi} \cdot \frac{r^{a+1}\sin\theta}{r^2 - 2r\cos\theta + 1}.$$

We need weighted Sobolev spaces on Ω. For all $m \in \mathbb{Z}$,

$$\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega) := H^m(\partial\Omega, g),$$

where the metric $g = (r^{-1}dr)^2$.

Layer Potentials on Plane Sectors

Proposition

The operators $\pm I + K_a := \pm I + M_{r^a} K M_{r^{-a}}$ *are invertible on* $\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega)$ *if and only if* $1 - (\widehat{\widetilde{K}}_a(\tau))^2 \neq 0$ *for all* $\tau \in \mathbb{R}$, *where* $\widehat{\widetilde{K}}_a$ *is the Mellin Transform of* $\widetilde{K}_a$.

We define a_θ by

$$a_\theta := \min\{\frac{\pi}{\theta}, \frac{\pi}{2\pi - \theta}\} = \begin{cases} \pi/(2\pi - \theta), & 0 < \theta < \pi, \\ \pi/\theta, & \pi < \theta < 2\pi. \end{cases}$$

Theorem

If $a \in (-a_\theta, a_\theta)$, *then* $\pm I + M_{r^a} K M_{r^{-a}}$ *is invertible on* $\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega)$, *that is,*

$$\pm I + K : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$$

are invertible.

Pseudodifferental Operators on Lie Groupoids

Definition

A *groupoid* is a small category $\mathcal{G}$ in which each arrow is invertible.

A groupoid $\mathcal{G}$ consists of two sets, a set of objects (or units) $\mathcal{G}_0$ and a set of arrows $\mathcal{G}_1$. Usually we shall denote the space of units of $\mathcal{G}$ by M and we shall identify $\mathcal{G}$ with $\mathcal{G}_1$. Each object of $\mathcal{G}$ can be identified with an arrow of $\mathcal{G}$, in that we have an injective map $u : M = \mathcal{G}_0 \to \mathcal{G}_1$, where $u(x)$ is the identity arrow of an object x. To each arrow $g \in \mathcal{G}$ we associate two units: its domain $d(g)$ and its range $r(g)$. The multiplication $\mu(g, h) = gh$ of two arrows $g, h \in \mathcal{G}$ is not always defined; it is defined exactly when $d(g) = r(h)$. The multiplication is associative. The inverse of an arrow is denoted by $g^{-1} = \iota(g)$. We can write

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "\mu"] & \mathcal{G}_1 \arrow[r, "\iota"] & \mathcal{G}_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "r"'] & \mathcal{G}_0 \arrow[r, "u"] & \mathcal{G}_1
\end{tikzcd}$$

(where $\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \subset \mathcal{G}_1 \times \mathcal{G}_1$ is the set of composable arrows).

Pseudodifferental Operators on Lie Groupoids

Definition

A *Lie groupoid* is a groupoid

$$\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, d, r, \mu, u, \iota)$$

such that $M := \mathcal{G}_0$ and $\mathcal{G}_1$ are smooth manifolds, possibly with corners, with M Hausdorff, the structural maps d, r, μ, u, and ι are smooth and the domain map d is a submersion (of manifolds with corners).

Example (Pair groupoid)

Let M be a smooth manifold (with or without corners). Let $\mathcal{G} = M \times M$ and $\mathcal{G}_0 = M$, with structure maps $d(m_1, m_2) = m_2$, $r(m_1, m_2) = m_1$, $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$, $u(m) = (m, m)$, and $\iota(m_1, m_2) = (m_2, m_1)$.

Example (Action groupoid)

Suppose that a Lie group G acts on the smooth manifold M from the right. The *transformation groupoid* over $M \times \{e\} \cong M$, denoted by $M \rtimes G$, is the set $M \times G$ with structure maps $d(m, g) = (m \cdot g, e)$, $r(m, g) = (m, e)$, $(m, g)(m \cdot g, h) = (m, gh)$, $u(m, e) = (m, e)$, and $\iota(m, g) = (m \cdot g, g^{-1})$.

Pseudodifferental Operators on Lie Groupoids

Definition

The space $\Psi^m(\mathcal{G})$ of *pseudodifferential operators of order m on a Lie groupoid $\mathcal{G}$* with units M consists of smooth families of pseudodifferential operators $P = (P_x)$, $x \in M$, with $P_x \in \Psi^m(\mathcal{G}_x)$, which are uniformly supported and right invariant.

For each $x \in M$, there is an interesting representation of $\Psi^\infty(\mathcal{G})$, the *regular representation* π_x on $C_c^\infty(\mathcal{G}_x)$, defined by $\pi_x(P) = P_x$. The *reduced C^*–norm* of P is defined by

$$\|P\|_r = \sup_{x \in M} \|\pi_x(P)\| = \sup_{x \in M} \|P_x\|,$$

and the *full norm* of P is defined by

$$\|P\| = \sup_{\rho} \|\rho(P)\|,$$

where ρ varies over all bounded representations of $\Psi^0(\mathcal{G})$ satisfying

$$\|\rho(P)\| \leqslant \|P\|_{L^1} \quad \text{for all} \quad P \in \Psi^{-\infty}(\mathcal{G}).$$

Definition

Let $\mathcal{G}$ be a Lie groupoid and $\Psi^\infty(\mathcal{G})$ be as above. We define $C^*(\mathcal{G})$ (respectively, $C_r^*(\mathcal{G})$) to be the completion of $\Psi^{-\infty}(\mathcal{G})$ in the norm $\|\cdot\|$ (respectively, $\|\cdot\|_r$). If $\|\cdot\|_r = \|\cdot\|$, that is, if $C^*(\mathcal{G}) \cong C_r^*(\mathcal{G})$, we call $\mathcal{G}$ *amenable*.

Pseudodifferental Operators on Lie Groupoids

For the case that the domain is a plane sector, let $\widetilde{\mathcal{H}} = [0,\infty] \rtimes \mathbb{R}^+$, where $\mathbb{R}^+ = (0,\infty)$ considered as a commutative group, and the action is just the multiplication.
So $\widetilde{\mathcal{H}}$ is an action groupoid. It is easy to see that

$$C^*(\widetilde{\mathcal{H}}) = C([0,\infty]) \rtimes \mathbb{R}^+.$$

Proposition

1. *Let* $a \in (-1,1)$. *We have* $M_{r^a} K M_{r^{-a}} \in C^*(\widetilde{\mathcal{H}}) \otimes M_2(\mathbb{C})$;
2. *For all* $k, l \in \mathbb{Z}$ *and* $a \in (-a_\theta, a_\theta)$, *the following mapping property holds:*

$$M_{r^a} K M_{r^{-a}} : \mathcal{K}^k_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^l_{\frac{1}{2}+a}(\partial\Omega).$$

Layer Potentials on Polygons

We still use Ω to denote a simply connected polygon.

Idea: Using a desingularization procedure, we construct a (natural) Lie groupoid $\mathcal{G}$ for Ω (gluing some Lie groupoids), and the double layer potential operator K (associated to Ω and the Laplace operator) can be identified with a "pseudodifferential operator" on $\mathcal{G}$.

We still use K to denote the double layer potential operator associated to Ω and the Laplace operator Δ. Let r_Ω be the smoothed distance to vertices of Ω and M_f be the multiplication operator by f.

Proposition

If $a \in (-1/2, 1/2)$, *then* $M_{r_\Omega^a} K M_{r_\Omega^{-a}} \in C^*(\mathcal{G})$.

Layer Potentials on Polygons

The following theorem is often used to prove certain operators are Fredholm.

Recall that π denotes the vector representation of $\Psi^{\infty}(\mathcal{G})$.

Theorem (Nistor and Lauter)

Suppose that $P = (P_m)$, $m \in M$, is a pseudodifferential operator on $\mathcal{G}$ or in $C^*(\mathcal{G})$. Then

$$\pi(P) \text{ is Fredholm } \iff P \text{ elliptic} + P_m \text{ invertible for all } m \in \partial M.$$

Recall that we have the following identification on Ω. For all $m \in \mathbb{Z}$,

$$\mathcal{K}^m_{\frac{1}{2}}(\partial\Omega) := H^m(\partial\Omega, g),$$

where the metric $g = (r_\Omega^{-1}\, dr_\Omega)^2$.

Layer Potentials on Polygons

Let $\theta_0 := \min\{\frac{\pi}{\theta_1}, \frac{\pi}{2\pi-\theta_1}, \frac{\pi}{\theta_2}, \frac{\pi}{2\pi-\theta_2}, \cdots, \frac{\pi}{\theta_n}, \frac{\pi}{2\pi-\theta_n}\}$. It is clear that $\frac{1}{2} < \theta_0 < 1$.

Proposition

Let Ω be a polygon on $\mathbb{R}^2$, and K be the double layer potential operator associated to Ω and the Laplace operator Δ. Then for $a \in (-\theta_0, 1/2)$, the operators

$$\pm I + K : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$$

are both Fredholm.

Our mail result is as follows:

Theorem (YQ and H. Li, 2017)

With the same notations as above, the operators

$$\pm I + K : \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega) \to \mathcal{K}^m_{\frac{1}{2}+a}(\partial\Omega)$$

are both isomorphisms for all $a \in (-\theta_0, 1/2)$.

Thanks for your attention!