Identification of inclusions in mfEIT using NPO

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Workshop on Neumann Poincaré Operator (NPO), Jeju S. Korea.
Feb. 8-10, 2018

Collaboration

The results presented in this talk, have been done in collaboration with

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Outline

- Identification of inclusions in mfEIT
  - The mathematical model
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- The forward problem
  - Spectral decomposition using NPO
  - Frequency dependence of the solution
- Uniqueness and stability estimates
  - Recovery of the frequency independent part
  - Identification of the inclusion
- Numerical Results

The Multifrequency Electrical Impedance Tomography

The mathematical model:
The mfEIT forward problem is to determine the potential $u(\cdot,\omega) \in H^1(\Omega) := \{v \in L^2(\Omega) : \nabla v \in L^2(\Omega)\}$, solution to

$$\begin{cases} -\nabla \cdot (\sigma(x,\omega)\nabla u(x,\omega)) = 0 \quad \text{in} \quad \Omega, \\ \sigma(x,\omega)\partial_{\nu_\Omega} u(x,\omega)(x) = f(x) \quad \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u(x,\omega) ds = 0, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain, $\omega \in [\underline{\omega}, \overline{\omega}]$ denotes the frequency, $\sigma(x,\omega) \in L^\infty(\Omega)$ is the conductivity distribution satisfying

$$\Re(\sigma(x,\omega)) > 0 \quad x \in \Omega, \ \omega \in [\underline{\omega}, \overline{\omega}],$$

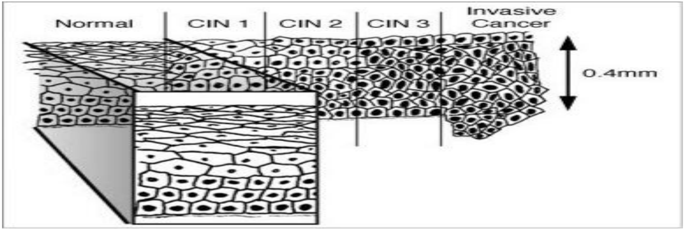
and $f \in H^{\frac{1}{2}}_\diamond(\partial\Omega) := \{g \in H^{\frac{1}{2}}(\partial\Omega) : \int_{\partial\Omega} g \, ds = 0\}$ is the input current.

▼ There exists a unique solution $u \in H^1(\Omega)$ to the forward problem (1).

The Multifrequency Electrical Impedance Tomography

Experimental observation:

- The conductivity $\sigma(x, \omega)$ of many biological tissues varies strongly with respect to the current frequency ω within certain frequency ranges. [1]
- Using homogenization techniques, effective biological tissue electrical properties have been derived. The frequency dependence of the conductivity reflects the tissue composition and physiology. [2]



[1] C. Gabriel, A. Peyman, and E.H. Grant. Electrical conductivity of tissue at frequencies below 1MHz. Phys. Med. Biol. 54 (09).

[2] H. Ammari, J. Garnier, L. Giovangigli, W. Jing, and J.K. Seo. Spectroscopic imaging of a dilute cell suspension. J. Math. Pures Appl. (16).

The Multifrequency Electrical Impedance Tomography

The mathematical model:

For simplicity we assume that σ takes the form

$$\sigma(x,\omega) = k_0 + (k(\omega) - k_0)\chi_D(x), \qquad \omega \in [\underline{\omega}, \overline{\omega}].$$

- $k_0 > 0$
  is the conductivity of the background.
- $\chi_D(x)$ is the characteristic
  function of a smooth inclusion $D \subset\subset \Omega$.
- $k(\omega) : [\underline{\omega}, \overline{\omega}] \to \mathbb{C} \setminus \mathbb{R}_-$,
  is a continuous not constant function
  satisfying $(\Re(k(\omega)) > 0)$. Then the set

$$\Sigma := \{k(\omega); \omega \in [\underline{\omega}, \overline{\omega}]\},$$

has an accumulation point in $\mathbb{C}^2$.
For example the empirical Drude model:

$$k(\omega) := \kappa_1 - \frac{\kappa_2}{\omega^2 + i\omega\kappa_3},$$

where $\kappa_p > 0$ are constants that depend on the biological tissue.

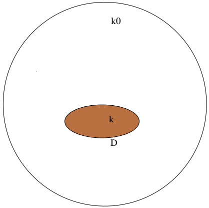


Figure: domain Ω.

The Multifrequency Electrical Impedance Tomography

mfEIT inverse problem:

The mfEIT inverse problem is to recover the shape and position of the inclusion D from measurements of the boundary voltages $u(x,\omega)$ on $\partial\Omega$ for $\omega \in [\underline{\omega}, \overline{\omega}]$, $0 \leq \underline{\omega} < \overline{\omega}$, that is

$$u(x,\omega)|_{\partial\Omega},\ \omega \in [\underline{\omega}, \overline{\omega}] \longrightarrow D$$

- A unique boundary voltage but for infinitely many frequencies: the experimental setting is easy to implement.
- In general measurements can be taken on an open subset $\Gamma \subset \partial\Omega$.

EIT is **cheap**, **noninvasive, portable,** *and allows continuous monitoring of the conductivity.*

Objective: to study the uniqueness and stability issues for inclusions within a given set $\mathfrak{D}$.

Assume $0 \in \Omega$, $b_1 = \mathrm{dist}(0, \partial\Omega)$ and $b_0 < b_1$. For $\delta > 0$ small enough, $\varsigma \in (0,1)$, and $m > 0$ large enough, define $\mathfrak{D}$ the set of inclusions:

$$\mathfrak{D} = \{x \in \mathbb{R}^d : |x| < \Upsilon(\hat{x}), \hat{x} = \frac{x}{|x|}\}; b_0 < \Upsilon(\hat{x}) < b_1 - \delta; \|\Upsilon\|_{C^{2,\varsigma}} \leq m.$$

The Electric Impedance Tomography

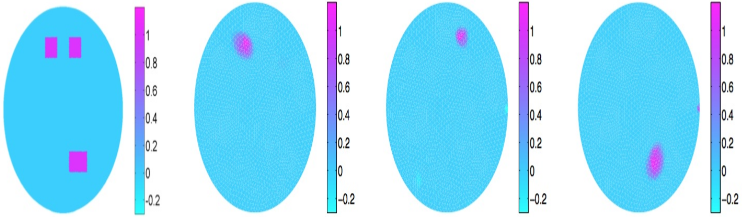
EIT with a single frequency.

k, k_0 are given, to find $\sigma(x) = k_0 + (k - k_0)\chi_D(x)$.

- Special geometries: Uniqueness for polygons (Friedmann and Isakov [88]), Uniqueness for convex polyhedron (Barcelo, Fabes and Seo [94], Uniqueness for balls (Kang and Seo [99]), Stability for disks (Fabes, Kang, Seo [99], Bonnetier-T-Tsou [2017]) .
- Local uniqueness in a class of smooth shapes by linearizing around a known shape (Alessendrini, Isakov and Powell [95])
- Small size inclusions: (Ammari and Kang [04]) Use the first polarization tensor to reconstruct ellipse equivalent shapes.
- Estimating the volume of the inclusion (Alessandrini, Kang, Rosset and Seo [98], Kang-Kim-Milton [11]).
- Uniqueness and stability estimates for the degenerate contrast cases ($k/k_0 \to 0, \infty$) (Bukhegeim-Cheng-Yamamoto [99], G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella [00]).

The Multifrequency Electrical Impedance Tomography

Multifrequency EIT



(a) true $\delta\sigma_k$s (b) recovered $\delta\sigma_1$ (c) recovered $\delta\sigma_2$ (d) recovered $\delta\sigma_3$

- G.S. Alberti, H. Ammari, B. Jing, and J.K. Seo, SIAM J. Imag. Sci. (2017).
- J. Jang and J.K. Seo, Phys. Meas. 36 (2015).
- R. J. Yerworth, R. H. Bayford, B. Brown, P. Milnes, M. Conway, and D. S. Holder, Physiol. Meas., 24(2), (2003).
- E. Malone, G. Sato dos Santos, D. Holder, and S. Arridge. IEEE Trans. Med. Imag., 33(2) (2014).
- S. Kim and A. Tamasan. Inverse Problems 30(3) (2014).

Main results

The mfEIT inverse problem has a unique solution within the class $\mathfrak{D}$, and we have the following stability estimates.

Theorem (AT[3])

Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Denote by u (resp. $\tilde{u}$) the solution of (1) with the inclusion D (resp. $\widetilde{D}$). Let

$$\varepsilon = \sup_{x\in\partial\Omega,\omega\in[\underline{\omega},\overline{\omega}]} |u - \tilde{u}|.$$

Then, there exist constants $C > 0$ and $\tau \in (0,1)$, such that the following estimate holds:

$$\left|D\Delta\widetilde{D}\right| \leq C\left(\frac{1}{\ln(\varepsilon^{-1})}\right)^{\tau}, \tag{2}$$

Here, Δ denotes the symmetric difference and the constants C and τ depend only on $f, \Omega, \mathfrak{D}$, and $\Sigma := \{k(\omega); \omega \in [\underline{\omega}, \overline{\omega}]\}$.

[3] H. Ammari, F. Triki, Identification of an inclusion in multifrequency electric impedance tomography, Communications in Partial Differential Equations, Volume 42, (2017)

Main results

Theorem

Assume that $d = 2$*, and let* D *and* $\widetilde{D}$ *be two analytic inclusions in* $\mathfrak{D}$*. Denote by* u *(resp.* $\tilde{u}$*) the solution of* (1) *with the inclusion* D *(resp.* $\widetilde{D}$*). Let*

$$\varepsilon = \sup_{x \in \partial\Omega, \omega \in [\underline{\omega}, \overline{\omega}]} |u - \tilde{u}|.$$

Then, there exist constants $C > 0$ *and* $\tau' \in (0, 1)$*, such that the following estimate*

$$\left| D \Delta \widetilde{D} \right| \leq C \varepsilon^{\tau'}, \tag{3}$$

holds. Here the constants C *and* τ' *depend only on* $f, \Omega, \mathfrak{D}$*, and* Σ*.*

Proofs

Sketch of the proof:

- Step 1: To understand the frequency dependence of the solution:

$$u(x,\omega) \quad = \quad k_0^{-1} u_0(x) + u_f(x, k(\omega)), \qquad x \in \partial\Omega$$

where $k \to u_f(x,k)$ is a meromorphic function on $\mathbb{C}$, and $u_0(x) \in H^1_\diamond(\Omega)$ satisfies

$$\begin{cases} \Delta v = 0 & \text{in} \quad D', \\ \nabla v = 0 & \text{in} \quad D, \\ \partial_{\nu_\Omega} v = f & \text{on} \quad \partial\Omega. \end{cases}$$

Tools: Spectral decomposition of the Poincaré variational operator.

- Step 2: To retrieve $u_0(x), x \in \partial\Omega$ from $u(x,\omega),\ \omega \in [\underline{\omega}, \overline{\omega}],\ x \in \partial\Omega$.
Tools: Rouché theorem; Unique continuation of meromorphic functions.

- Step 3: To reconstruct D from the single Cauchy data $(u_0|_{\partial\Omega}, f)$.
Tools: Unique continuation and the fact that the tangential derivative of u_0 is zero on ∂D.

The variational Poincaré operator

Let $H^1_\diamond(\Omega)$ be the space of functions $\{v \in H^1(\Omega);\ \int_{\partial\Omega} vds = 0\}$ endowed with the norm with finite energy semi-norm

$$\|u\|_{H^1_\diamond} = \int_\Omega |\nabla u|^2 dx.$$

For $u \in H^1_\diamond(\Omega)$, we infer from the Riesz theorem that there exists a unique function $Tu \in H^1_\diamond(\Omega)$ such that for all $v \in H^1_\diamond(\Omega)$,

$$\int_\Omega \nabla Tu \cdot \nabla v dx = \int_D \nabla u \cdot \nabla v dx.$$

The variational Poincaré operator $T : H^1_\diamond(\Omega) \to H^1_\diamond(\Omega)$ is self-adjoint and bounded with norm $\|T\| \leq 1$.

The spectral problem for T reads as: Find $(\lambda, w) \in \mathbb{R} \times H^1_\diamond(\Omega)$, $w \neq 0$ such that $\forall v \in H^1_\diamond(\Omega)$,

$$\lambda \int_\Omega \nabla w \cdot \nabla v dx = \int_D \nabla w \cdot \nabla v dx.$$

Integrating by parts, one obtains that any eigenfunction w is harmonic in $\Omega \setminus \partial D$, and satisfies

$$w|^+_{\partial D} = w|^-_{\partial D}, \qquad \partial_{\nu_D} w|^+_{\partial D} = (1 - \frac{1}{\lambda})\partial_{\nu_D} w|^-_{\partial D}, \qquad \partial_{\nu_\Omega} w = 0,$$

The variational Poincaré operator

Let $\mathfrak{H}_\diamond$ the space of harmonic functions in D and $D' := \Omega \setminus \overline{D}$, with zero mean $\int_{\partial\Omega} u ds(x) = 0$, and zero normal derivative $\partial_{\nu_\Omega} u = 0$ on $\partial\Omega$.

Theorem

The variational Poincaré operator $T : \mathfrak{H}_\diamond \to \mathfrak{H}_\diamond$ has the following decomposition

$$T = \frac{1}{2} I + K,$$

where K is a compact self-adjoint operator.

The proof is based on layer potential techniques. One can show that $K : \mathfrak{H}_\diamond \to \mathfrak{H}_\diamond$ is compact if and only if the Neumann-Poincaré operator $\mathcal{K}_D^* : H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$, defined by

$$\mathcal{K}_D^* \varphi(x) = \frac{1}{|S^{d-1}|} \int_{\partial D} \frac{\nu_D(x) \cdot (x-y)}{|x-y|^d} \varphi(y) ds(y),$$

is compact (which is the case for $C^{1,\varsigma}$ inclusions).

The variational Poincaré operator

We denote by $(\lambda_n^-)_{n\geq 1}$ the eigenvalues of T repeated according to their multiplicity, and ordered as follows

$$0 < \lambda_1^- \leq \lambda_2^- \leq \cdots < \lambda_\infty^+ = \frac{1}{2},$$

in $(0, 1/2]$ and, similarly,

$$1 > \lambda_1^+ \geq \lambda_2^+ \geq \cdots > \lambda_\infty^+ = \frac{1}{2}.$$

the eigenvalues in $[1/2, 1)$. The eigenvalue $1/2$ is the unique accumulation point of the spectrum.

In dimension two, if $\Omega = \mathbb{R}^2$, we have $\lambda_j^+ = 1 - \lambda_j^-$. [4]

The eigenfunctions $w_n^\pm$ form an orthonormal basis of $\mathfrak{H}_\diamond$.

[4] D. Khavinson, M. Putinar, H.S. Shapiro, Poincaré's variational problem in potential theory. Arch. Ration. Mech. Anal. 185, 143?184 (2007)

The variational Poincaré operator

Let $w_n^{\pm}$, $n \geq 1$ be the eigenfunctions associated to the eigenvalues $(\lambda_n^-)_{n\geq 1}$. Then

$$\begin{aligned}
\lambda_1^- &= \min_{0\neq w\in\mathfrak{H}_\diamond} \frac{\int_D |\nabla w(x)|^2 dx}{\int_\Omega |\nabla w(x)|^2 dx}, \\
\lambda_n^- &= \min_{\substack{w\in\mathfrak{H}_\diamond\setminus\{0\} \\ u\perp w_1^-,\dots,w_{n-1}^-}} \frac{\int_D |\nabla w|^2\, dx}{\int_\Omega |\nabla w|^2\, dx},
\end{aligned}$$

and similarly

$$\begin{aligned}
\lambda_1^+ &= \max_{0\neq w\in\mathfrak{H}_\diamond} \frac{\int_D |\nabla w(x)|^2 dx}{\int_\Omega |\nabla w(x)|^2 dx}, \\
\lambda_n^+ &= \max_{\substack{w\in\mathfrak{H}_\diamond\setminus\{0\} \\ w\perp w_1^+,\dots,w_{n-1}^+}} \frac{\int_D |\nabla w|^2\, dx}{\int_\Omega |\nabla w|^2\, dx}.
\end{aligned}$$

Spectral decomposition

Theorem

Let $u(x,\omega)$ be the unique solution to the system (1). Then the following decomposition holds:

$$u(x,\omega) = k_0^{-1}u_0(x) + \sum_{n=1}^{\infty} \frac{\int_{\partial\Omega} f(z)w_n^{\pm}(z)ds(z)}{k_0 + \lambda_n^{\pm}(k(\omega)-k_0)} w_n^{\pm}(x), \quad x \in \Omega, \tag{4}$$

$$= k_0^{-1}u_0(x) + u_f(k(\omega),x), \quad x \in \Omega, \tag{5}$$

where $u_0(x) \in H^1_\diamond(\Omega)$ depends only on f and D, and is the unique solution to

$$\begin{cases} \Delta v = 0 & in \quad D', \\ \nabla v = 0 & in \quad D, \\ \partial_{\nu_\Omega} v = f & on \quad \partial\Omega. \end{cases} \tag{6}$$

Retrieval of the frequency dependent part

$$u(x,\omega) = k_0^{-1} u_0(x) + \sum_{n=1}^{\infty} \frac{(\lambda_n^{\pm})^{-1} \int_{\partial\Omega} f(z) w_n^{\pm}(z) ds(z)}{k(\omega) - k_n^{\pm}} w_n^{\pm}(x), \quad x \in \partial\Omega,$$

where $k_n^{\pm} := k_0(1 - \frac{1}{\lambda_n^{\pm}})$, and they can be ordered as follows:

$$k_1^- \leq k_2^- \leq \cdots < -k_0 < \cdots \leq k_2^+ \leq k_1^+ < 0.$$

Lemma

There exists a constant $\hat{\delta} > 0$ depending only on $\mathfrak{D}$ such that

$$k_n^{\pm} \geq -\hat{\delta}^{-1}, \qquad \forall n \geq 1.$$

A forward computation shows $r_D := \inf_{x \in \partial D} x \cdot \nu_D(x) > C_{\mathfrak{D}} > 0$. Using Rellich's identities[5] we obtain ($\Omega = \mathbb{R}^2$):

$$-\infty < -1 - \left(\frac{r_D + 2}{r_D}\right)^2 \leq k_1^-,$$

In [Bonnetier-T13], it was shown that $k_1 \to -\infty$ for touching inclusions.

[5] H. Ammari and J.K. Seo. An accurate formula for the reconstruction of conductivity inhomogeneities. Adv. Appl. Math., 30 (2003).

Retrieval of the frequency dependent part

To apply the unique continuation for holomorphic complex functions we also need the following bounds.

Theorem

Let D be an inclusion in $\mathfrak{D}$. Then there exists a constant $C = C(\mathfrak{D}, \Omega, k_0) > 0$ such that

$$\|u_f(x,k)\|_{C^0(\partial\Omega)} \leq C\left(\left(1 + \frac{1}{dist(k,[k_1^-,0])}\right)\right)\|f(x)\|_{H^{\frac{1}{2}}(\partial\Omega)}. \qquad (7)$$

The constant C tends to $+\infty$ as $\hat{\delta}$ tends to zero.

Retrieval of the frequency dependent part

$$
\begin{aligned}
u(x,\omega) &= k_0^{-1}u_0(x) + \sum_{n=1}^{\infty} \frac{\int_{\partial\Omega} f(z) w_n^{\pm}(z)ds(z)}{k_0 + \lambda_n^{\pm}(k(\omega) - k_0)} w_n^{\pm}(x), \quad x \in \partial\Omega, \\
&= k_0^{-1}u_0(x) + u_f(k(\omega), x), \quad x \in \partial\Omega,
\end{aligned}
$$

Theorem

Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Denote by u (resp. $\tilde{u}$), the solution of (1) with inclusion D (resp. $\widetilde{D}$). Let

$$
\varepsilon = \sup_{x\in\partial\Omega, \omega\in(\underline{\omega},\overline{\omega})} |u - \tilde{u}|.
$$

Then, there exists a constant $\kappa > 0$, that depends only on $\Omega, \mathfrak{D}, k_0$, and Σ, such that

$$
\sup_{x\in\partial\Omega, \omega\in(\underline{\omega},\overline{\omega})} |u_f - \tilde{u}_f| \leq C\varepsilon^{\kappa}, \tag{8}
$$

where the constant $C > 0$ depends only on $f, \Omega, \mathfrak{D}$, and Σ.

Retrieval of the frequency dependent part

For $x \in \partial\Omega$ fixed
$$\alpha(k) = k_0^{-1} u_0(x) + u_f(x, k),$$
is a meromorphic function with poles $(k_n^{\pm})_{n \geq 1}$.
Similarly for $x \in \partial\Omega$ fixed, we have
$$\tilde{\alpha}(k) = k_0^{-1} \tilde{u}_0(x) + \tilde{u}_f(x, k),$$
is a meromorphic function with poles $(\tilde{k}_n^{\pm})_{n \geq 1}$ (the plasmonic resonances of the inclusion $\widetilde{D}$).

Let $\mathcal{C}_+$ be a Jordan complex contour with interior $\mathring{\mathcal{C}}_+$ that contains $[-\hat{\delta}^{-1}, 0) \cup \overline{\Sigma}$.

Let $\mathcal{C}_-$ be a Jordan complex contour in $\mathring{\mathcal{C}}_+$, with interior $\mathring{\mathcal{C}}_-$ that contains $[-\hat{\delta}^{-1}, 0]$ and does not intersect Σ, that is, $\mathring{\mathcal{C}}_- \cap \Sigma = \emptyset$.

Finally, let $\mathcal{C}$ be a Jordan complex contour in $\mathring{\mathcal{C}}_+ \setminus \mathring{\mathcal{C}}_-$ such that $[-\hat{\delta}^{-1}, 0] \subset \mathring{\mathcal{C}}$, and $\mathring{\mathcal{C}} \cap \Sigma = \emptyset$.

Retrieval of the frequency dependent part

Let ω be a fixed frequency in $(\underline{\omega}, \overline{\omega})$. Since the poles $(k_n^{\pm})_{n\geq 1}, (\tilde{k}_n^{\pm})_{n\geq 1}$ are inside $\overset{\circ}{\mathcal{C}}$, and $k(\omega)$ lies in the exterior of $\mathcal{C}$, we have

$$u_f(x, k(\omega)) - \tilde{u}_f(x, k(\omega)) = \frac{1}{2i\pi} \int_{\mathcal{C}} \frac{\alpha(k) - \tilde{\alpha}(k)}{k - k(\omega)} dk.$$

Consequently,

$$|u_f(x, k(\omega)) - \tilde{u}_f(x, k(\omega))| \leq \frac{1}{\mathrm{dist}(\Sigma, \mathcal{C})} \|\alpha(k) - \tilde{\alpha}(k)\|_{L^\infty(\mathcal{C})}. \tag{9}$$

Now, define $w(z)$ to be the harmonic measure of $\overline{\Sigma}$ in $\overset{\circ}{\mathcal{C}}_+ \setminus \overline{\overset{\circ}{\mathcal{C}}_-}$, which is holomorphic in $\overset{\circ}{\mathcal{C}}_+ \setminus \overline{\overset{\circ}{\mathcal{C}}_-}$ and statifies $w(z) = 1$ on $\overline{\Sigma}$, $w(z) = 0$ on $\mathcal{C}_- \cup \mathcal{C}_+$.
Then the two-constants theorem implies

$$|\alpha(k) - \tilde{\alpha}(k)| \leq M^{1-w(k)} \varepsilon^{w(k)},$$

for all k in $\overset{\circ}{\mathcal{C}}_+ \setminus \overset{\circ}{\mathcal{C}}_-$ where $M = \max_{k \in \overset{\circ}{\mathcal{C}}_+ \setminus \overset{\circ}{\mathcal{C}}_-} (|\alpha(k)| + |\tilde{\alpha}(k)|)$.

Retrieval of the frequency independent part

Corollary

Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Denote u (resp. $\tilde{u}$), the solution of (1) *with inclusion D (resp. $\widetilde{D}$). Let*

$$\varepsilon = \sup_{x \in \partial\Omega, \omega \in (\underline{\omega}, \overline{\omega})} |u - \tilde{u}|.$$

Then, there exists a constant $\kappa > 0$, that depends only on $\Omega, \mathfrak{D}$ and Σ, such that

$$\sup_{x \in \partial\Omega, \omega \in (\underline{\omega}, \overline{\omega})} |u_0 - \tilde{u}_0| \leq C\varepsilon^{\kappa}, \tag{10}$$

where the constant $C > 0$ only depends on $f, \Omega, \mathfrak{D}$ and Σ.

Retrieval of the Cauchy data

Lemma (BCY[6])

Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let u_0 (resp. $\tilde{u}_0$) be the solution in $H^1_\diamond(\Omega)$ of (6) with inclusion D (resp. $\widetilde{D}$), and assume that

$$0 < \varepsilon = \sup_{x \in \partial\Omega} |u_0 - \tilde{u}_0| < 1.$$

Then, there exist constants $C > 0$ and $\mu > 0$, such that the following estimate holds:

$$\|u_0 - \tilde{u}_0\|_{C^0(\partial(D \cup \widetilde{D}))} \leq C \left(\frac{1}{\ln(\varepsilon^{-1})} \right)^{\mu}. \qquad (11)$$

Here, the constants C and μ depend only on f, Ω, and $\mathfrak{D}$.
If, in addition $d = 2$, and the inclusions D and $\widetilde{D}$ are analytic, then we have

$$\|u_0 - \tilde{u}_0\|_{C^0(\partial(D \cup \widetilde{D}))} \leq C \varepsilon^{\mu'}, \qquad (12)$$

where the constants C and μ' depend only on f, Ω, and $\mathfrak{D}$.

[6] A. L. Bukhgeim, J. Cheng, and M. Yamamoto. Stability for an inverse boundary problem of determining a part of boundary, Inverse Problems, 14 (1999).

Reconstruction of the inclusion

Theorem

Let D and $\widetilde{D}$ be two inclusions in $\mathfrak{D}$. Let

$$\varepsilon = \sup_{x\in\partial\Omega,\omega\in(\underline{\omega},\overline{\omega})} \left|u_0 - \tilde{u}_0\right|.$$

Then, there exist constants $C > 0$ and $\tau \in (0,1)$, such that the following estimate holds:

$$\left|D\Delta\widetilde{D}\right| \leq C\varepsilon^{\tau}, \tag{13}$$

Here, Δ denotes the symmetric difference and the constants C and τ depend only on $f, \Omega, \mathfrak{D}$, and $\Sigma := \{k(\omega); \omega \in [\underline{\omega}, \overline{\omega}]\}$.

Quantification of how $u_0(x)$ is close to its constant value in D when $x \to \partial D$[7].

[7] G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze 29 (2000).

Numerical results

We have two principal steps[8]:

- Retrieve $u_0(x),\ x \in \partial\Omega$ from the spectral decomposition for $\omega \in (\omega_p)_{1 \le p \le N_\omega}$.

$$\lambda_n^{\pm} \approx \frac{1}{2}, \quad n \ge N_\lambda + 1, \ (\lambda_n \to 1/2).$$

$$u(x,\omega_p) \approx k_0^{-1} u_0(x) + \sum_{n=1}^{N_\lambda} \frac{1}{k_0 + \lambda_n^{\pm}(k(\omega_p) - k_0)} v_n^{\pm}(x) + \frac{2}{k(\omega_p) + k_0} v_{N_\lambda+1}(x).$$

- Reconstruct D from Cauchy data $(u_0^{(i)}, f_i),\ i = 1, 2$, on $\partial\Omega$:

$$J(D) = \frac{1}{2} \int_{\partial\Omega} \sum_{i=1}^{2} |u_0^{(i)} - u_{meas}^{(i)}|^2 ds(x),$$

where $u_{meas}^{(i)}$ are the measured Dirichlet data corresponding to the i-th Neumann data $f_i = x_i,\ i = 1, 2$.

[8] H Ammari, F Triki, C-H Tsou, Numerical determination of anomalies in multifrequency electrical impedance tomography arXiv preprint arXiv:1704.04878 (2017).

Step 1: Recovery of u_0

We further assume that $k_0 = 1$.

Observation: Since $1/2$ is the unique accumulation point of the eigenvalues $(\lambda_n^{\pm})_{n\in\mathbb{N}^*}$, we only consider the N_f first eigenvalues as unknowns, and we approximate the others eigenvalues by $1/2$. In fact it has been shown in [MiyanishiSuzuki15] that if D is C^β with $\beta \geq 2$ then for any $\alpha > -2\beta + 3$, we have

$$|\lambda_n^{\pm} - 1/2| = o(n^\alpha), \to n \to +\infty.$$

If the boundary of D is C^∞ smooth, then the eigenvalues decay faster than any power.

Recently the authors in [AndoKangMiyanishi16][9] have shown the exponential convergence of the eigenvalues in the case of analytic inclusions.

Approximation: $\lambda_n^{\pm} \approx \frac{1}{2}, \quad n \geq N_f + 1,$
where N_f depends only the smoothness of the inclusion D

[9] K Ando, H Kang, Y Miyanishi, Exponential decay estimates of the eigenvalues for the Neumann-Poincaré operator on analytic boundaries in two dimensions, arXiv preprint arXiv:1606.01483, 2016

Step 1: Recovery of u_0

We consider the N_ω frequencies of measurements $\omega_1, \omega_2, \cdots, \omega_{N_\omega}$. We estimate arbitrarily the N_λ first eigenvalues, and we approximate the others eigenvalues by $\frac{1}{2}$. That means, we make the following approximation, for $x \in \Omega$, $1 \leq p \leq N_\omega$,

$$
\begin{aligned}
u(x, \omega_p) \approx u_{approx}(x, \omega_p) = & u_0(x) + \sum_{n=1}^{N_\lambda} \frac{1}{1 + \lambda_n^{\pm}(k(\omega_p) - 1)} v_n^{\pm}(x) \\
& + \frac{2}{k(\omega_p) + 1} v_{N_\lambda + 1}(x),
\end{aligned}
\tag{14}
$$

where

$$
v_n^{\pm}(x) = \int_{\partial\Omega} f(z) w_n^{\pm}(z) ds(z) w_n^{\pm}(x),
$$

and

$$
v_{N_\lambda+1}(x) = \sum_{n > N_\lambda} \int_{\partial\Omega} f(z) w_n^{\pm}(z) ds(z) w_n^{\pm}(x).
$$

Step 1: Recovery of u_0

Using a simple integration by parts, we get,

$$u_{approx}(x,\omega_p) = \frac{k(\omega_p)-1}{(k(\omega_p)+1)} u_0(x) + \frac{2}{k(\omega_p)+1}\mathfrak{f}(x) + \sum_{n=1}^{N_\lambda}\left(\frac{1}{1+\lambda_n^{\pm}(k(\omega_p)-1)} - \frac{2}{k(\omega_p)+1}\right)v_n^{\pm}(x), \quad (15)$$

where $\mathfrak{f}$ is the solution to the system

$$\begin{cases} \triangle\mathfrak{f} = 0 & \text{in } \Omega, \\ \partial_\nu \mathfrak{f} = f & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \mathfrak{f} ds = 0. \end{cases} \quad (16)$$

In [AlbertiAmmariJingSeo2017], they considered the following approximation

$$u_{approx}(x,\omega_p) = \mathfrak{f}(x) + U(x,\omega_p),$$

which is valid if $k(\omega) \sim 1$ or $|D|$ is small.

Step 1: Recovery of u_0

Our optimization algorithm:

1. Discretization of $\partial\Omega$, $(x_j)_{1\leq j\leq N_x}$.
2. Give an apriori estimation of the eigenvalues $\widetilde{\lambda_n^{\pm}} \in [0,1]$ for $1 \leq n \leq N_\lambda$.
3. Minimizing the functional

$$J_m(U_0, V_1^{\pm}, \cdots, V_{N_\lambda}^{\pm}) := \frac{1}{2}\sum_{p=1}^{N_\omega}\sum_{j=1}^{N_x} |u(x_j,\omega_p) - u_{approx}(x_j,\omega_p)|^2. \tag{17}$$

Step 2: Recovery of D

The second step of our algorithm consists in determining the inclusion D from the knowledge of u_0 on $\partial\Omega$.

We assume that our domain D is star shaped and its boundary ∂D can be described by the Fourier series:

$$\partial D = \{X_0 + r(\theta)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} | \theta \in [0; 2\pi)\}, \; r = \sum_{n=-N}^{N} c_n f_n, \qquad (18)$$

where $C = \begin{pmatrix} c_{-N} \\ c_{-N+1} \\ \vdots \\ c_N \end{pmatrix} \in \mathbb{R}^{2N+1}$, $f_n(\theta) = \cos(n\theta)$ for $0 \leq n \leq N$ and $f_n(\theta) = \sin(n\theta)$ for $-N \leq n < 0$.

Step 2: Shape derivative

Let D_ε be the perturbed domain, which is given by

$$\partial D_\varepsilon = \{\tilde{x} : \tilde{x} = x + \varepsilon h(x)\nu(x), x \in \partial D\}, \tag{19}$$

where $h \in \mathcal{C}^1(\partial D)$ and ν denote the unit outward normal vector. We denote by u_ε the solution to (6) with D_ε instead of D.

Theorem

For $x \in \partial\Omega$,

$$u_\varepsilon(x) = u_0(x) + \varepsilon u_h(x) + O(\varepsilon^2), \tag{20}$$

where u_h satisfy the following equation,

$$\begin{cases} \triangle u_h = 0 & \text{in } D \cup (\Omega \setminus \overline{D}), \\ \frac{\partial u_h}{\partial \nu}\big|_- = 0 & \text{on } \partial D, \\ u_h\big|_+ - u_h\big|_- = -h\frac{\partial u_0}{\partial \nu}\big|_+ & \text{on } \partial D, \\ \frac{\partial u_h}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_h d\sigma = 0. \end{cases} \tag{21}$$

Step 2: Recovery of D

Our algorithm consists to minimizing the functional

$$J(D) = \frac{1}{2} \int_{\partial\Omega} \sum_{i=1}^{P} |u_{0,D} - u_0^{(i)}|^2 d\sigma,$$

where $u_{0,D}$ is the solution to the inverse boundary problem, and $u_0^{(i)}$ means the measurement of $u_0|_{\partial\Omega}$ corresponding to the i-th Neumann data, which is obtained from the first step.

1. Choose an initial domain D_0.
2. For each iteration, $i > 0$:
   - 2.1 Calculate u_i solution to (1) , associated to the domain D_i for which the boundary ∂D_i given by (18).
   - 2.2 Calculate the shape derivatives $\frac{\partial J}{\partial x_1}$, $\frac{\partial J}{\partial x_2}$ and $\frac{\partial J}{\partial c_n}$ for all $-N \leq n \leq N$.
   - 2.3 Update the parameters of the domain $X_{i+1} = X_i - \alpha \nabla_{X_0} J(X_i, C_i)$ and $C_{i+1} = C_i - \alpha \nabla_C J(X_i, C_i)$ with $\alpha > 0$.
   - 2.4 If the updated domain is not entirely in Ω or if the boundaries intersect, reduce the size of α.
3. When $J(X_i, C_i)$ becomes smaller than a fixed threshold, we stop.

Numerical simulations

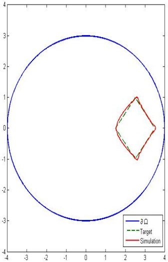
- We use FreeFem++ [10] for our numerical experiments.
- Ω is given by
  - a centered ellipse defined by the equation: $\frac{x_1^2}{4^2} + \frac{x_2^2}{3^2} \leq 1$.
  - $\partial\Omega = \{[4 + 0.8(\cos\theta + \sin\theta) - (\cos 2\theta + \sin 2\theta)](\cos\theta, \sin\theta), \theta \in [0, 2\pi)\}$.
- We use two linearly independent Neumann data: $f_1 = \langle e_1, \nu\rangle$ and $f_2 = \langle e_2, \nu\rangle$, where (e_1, e_2) is the canonical base of $\mathbb{R}^2$.
- Only the first two eigenvalues in the representation formula are taken into consideration, and we take $\widetilde{\lambda_1^+} = \frac{3}{4}$, $\widetilde{\lambda_1^-} = \frac{1}{4}$ and the others equal to $\frac{1}{2}$.

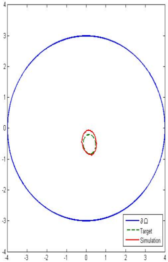
[10] F. Hecht, New development in FreeFem++ - Journal of Numerical Mathematics (2012)

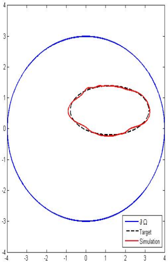
Numerical simulations

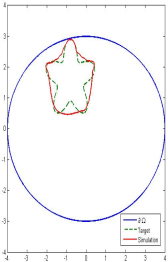
- In the algorithm to reconstruct u_0 and the conductivity profile, the initial guess of u_0 is the function $\mathfrak{f}$.
- The initial estimation of domain D is a centered disk with a radius $\frac{1}{2}$.
- We consider the first 15 Fourier coefficients: $N = 15$.
- We use P_1 finite elements for the numerical resolution of the PDEs.
- At each iteration, we remesh the domain corresponding to the updated position and shape of the inclusion.
- The algorithms stop if $J < 10^{-5}$ or the number of iterations exceeds 500. In all the presented cases we have executed 500 iterations.

Numerical results









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AIP, Conference July 2019 in Grenoble!

Thanks!!