Mathematical Studies of Extraordinary Field Enhancement in Subwavelength Slit Structures

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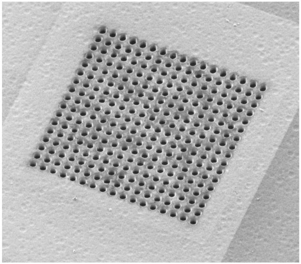
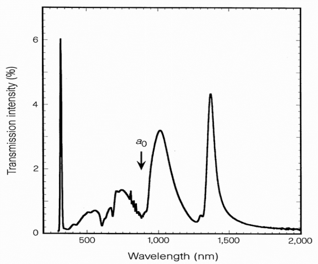
Joint work with Junshan Lin, Auburn University, USA

Extraordinary Optical Transmission Through a Small Hole Array

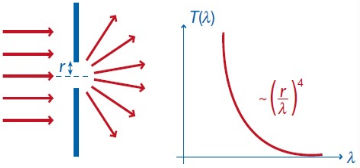
T. W. Ebbessen *et al*, Nature (1998)

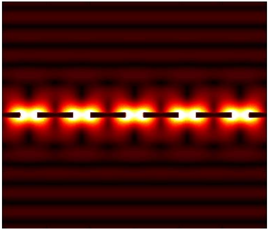
Size of each hole: 150 nm, metal thickness: 300 nm, skin depth: 30nm





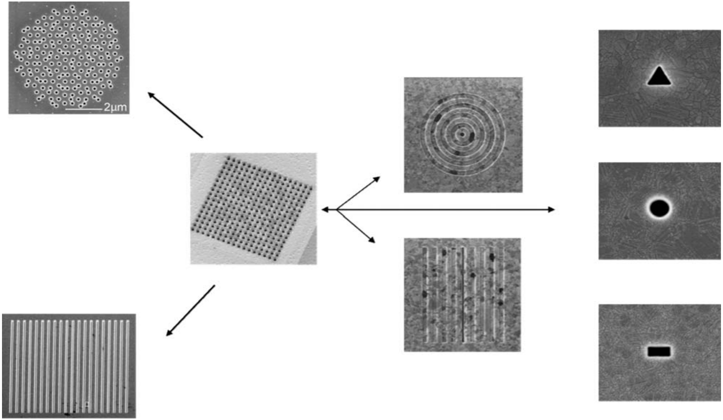
Classical Bethe theory for diffraction by a small hole





Subsequent Development in Extraordinary Optical Field Enhancement

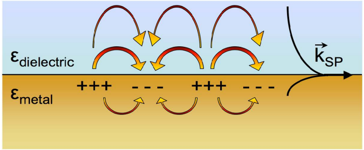
F. J. Garcia-Vidal *et al*, Rev. Mod. Phy. (2010)
S. Rodrigo, F. León-Pérez, L. Martín-Moreno, Proceedings of the IEEE (2016)

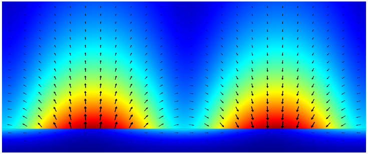


Applications: Near-field optical imaging, biosensing, novel optical devices....

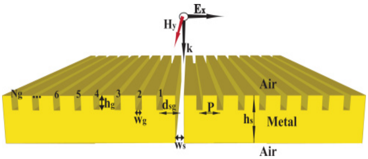
Possible Enhancement Mechanisms

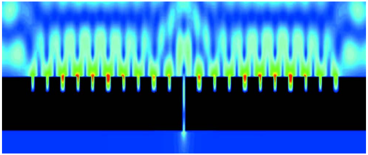
- Surface plasmonic resonances in noble metals





- Non-plasmonic resonances (e.g., resonances induced by the geometry of the structure)

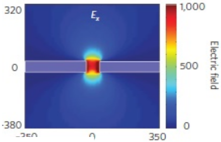




- Non-resonant enhancement,....

- There has been a long debate on the interpretation of enhancement effects. For instance, surface plasmonic resonances strengthen or inhibit the enhancement? interplay between different enhancement mechanisms?
- Other questions: How large is the field enhancement and at what frequencies?
- Quantitative analysis of the field enhancement would be desirable!
- Efficient numerical modeling techniques, optimal design for better performance.

Focus on a Prototype Structure: Narrow Slits



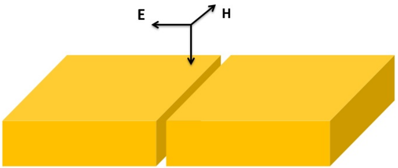
Field enhancement for slit structures in perfect conducting (PEC) metals:

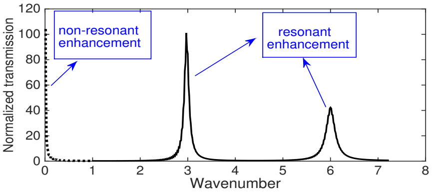
- Single slit: resonant and non-resonant enhancement effects.
- An array of slits: resonant and non-resonant enhancement effects, surface bound state, “surface plasmon" and total transmission phenomenon.

Related work:

- E. Bonnetier and F. Triki (2010): Resonances for a subwavelength cavity.
- High transmission for the periodic structures: G. Bouchitté, B. Schweizer, G. Kriegsmann, and many others in physics literatures.

Electromagnetic Field Enhancement in a Single PEC Slit





Transmission with metal thickness = 1, gap size = 0.02.

- Resonant effect

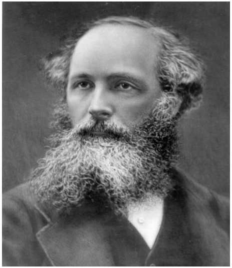
  Y. Takakura (2001), J. Sambles *et al* (2002), F. Garcia-Vidal, *et al* (2004), R. Gordon (2006) $\cdots$

- Non-resonant effect

  Experiments: D-S. Kim (2009), S-H. Oh (2014)

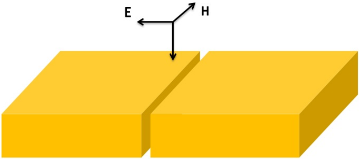
Answer the following questions:

- What are the resonant frequencies?
- Can one characterize the wave modes at resonant frequencies?
- What induces the enhancement at non-resonant frequencies?



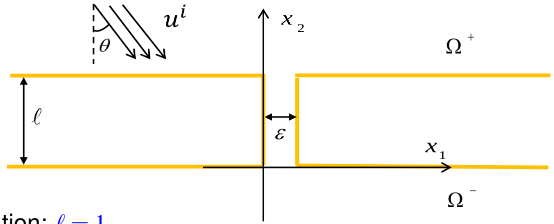
- τ and μ: electric permittivity and magnetic permeability.
- Time-harmonic Maxwell's equations (with $e^{-i\omega t}$ dependence):

$$\begin{cases} \nabla \times E = i\omega\mu H, & \text{(Faraday's law)} \\ \nabla \cdot (\tau E) = 0, & \text{(Gauss's law)} \\ \nabla \times H = -i\omega\tau E, & \text{(Ampère's law)} \\ \nabla \cdot (\mu H) = 0. \end{cases}$$



- Transverse magnetic polarization: $H = (0, 0, u)$, then in a homogeneous medium, u satisfies the Helmholtz equation

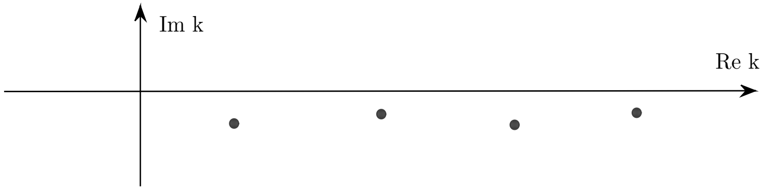
$$\Delta u + \left(\frac{\omega}{c}\right)^2 u = 0.$$



- Normalization: $\ell = 1$.
- The exterior domain: $\Omega_\varepsilon = \Omega_+ \cup \Omega_- \cup S_\varepsilon$.
- TM polarization: the incident magnetic field $H^i = (0,0,u^i)$, where $u^i = e^{ikd\cdot x}$, $k = \omega/c$.
- The total field $u_\varepsilon = u^i + u^r + u^s_\varepsilon$ in Ω^+, and $u_\varepsilon = u^s_\varepsilon$ (transmitted wave) in Ω^-.
- The scattering problem:

$$\begin{cases} \Delta u_\varepsilon + k^2 u_\varepsilon = 0 \qquad \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon. \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^s_\varepsilon}{\partial r} - iku^s_\varepsilon\right) = 0, \quad r = |x|. \end{cases}$$

Scattering Resonances and Field Enhancement

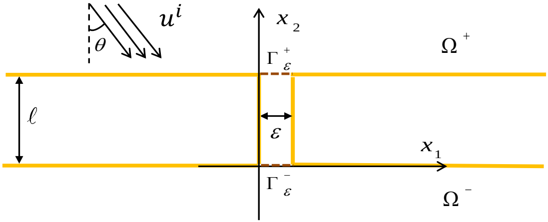


- Fact: The scattering problem attains a unique solution if Im $k \geq 0$.

Defintion

The *scattering resonances* are the poles of the resolvent associated with the scattering operator when continued meromorphically to the whole complex plane.

- Field enhancement at resonant frequencies: $O\left(\frac{1}{|k-k_{res}|}\right)$.



- Integral equation formulation:

$$\begin{cases} \displaystyle\int_{\Gamma_\varepsilon^+} g^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = -(u^i+u^r), \quad \text{on } \Gamma_\varepsilon^+, \\ \displaystyle\int_{\Gamma_\varepsilon^-} g^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = 0, \quad \text{on } \Gamma_\varepsilon^-. \end{cases}$$

- Boundary integral equations after scaling ($x_1=\varepsilon X,\ y_1=\varepsilon Y,\ X,Y\in(0,1)$):

$$\begin{bmatrix} T^e+T^i & \tilde{T}^i \\ \tilde{T}^i & T^e+T^i \end{bmatrix}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}.$$

where T^e, T^i, and $\tilde{T}^i$ are the integral operators with kernels G_ε^e, G_ε^i and $\tilde{G}_\varepsilon^i$, $\varphi_1(X) := -\partial_\nu u_\varepsilon(\varepsilon X,1)$, and $\varphi_2(X) := -\partial_\nu u_\varepsilon(\varepsilon X,0)$.

- Asymptotic expansions of the kernels:

$$
\begin{aligned}
G^e_\varepsilon(X,Y) &= \frac{1}{\pi}[\ln\varepsilon + \ln k + \gamma_0] + \frac{1}{\pi}\ln|X-Y| + O((\varepsilon|X-Y|)^2\ln(\varepsilon|X-Y|);\\
G^i_\varepsilon(X,Y) &= \frac{\cot k}{k\varepsilon} + \frac{2\ln 2}{\pi} + \frac{1}{\pi}\left[\ln\left(\left|\sin\left(\frac{\pi(X+Y)}{2}\right)\right|\right) + \ln\left(\left|\sin\left(\frac{\pi(X-Y)}{2}\right)\right|\right)\right]\\
&\quad + O(k^2\varepsilon^2);\\
\tilde{G}^i_\varepsilon(X,Y) &= \frac{1}{(k\sin k)\varepsilon} + O\left(e^{-1/\varepsilon}\right).
\end{aligned}
$$

- Asymptotic expansions of the integral operators:

$$
\begin{bmatrix} T^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T^e + T^i \end{bmatrix} = \begin{bmatrix} \beta & \tilde{\beta} \\ \tilde{\beta} & \beta \end{bmatrix} P + K\mathbb{I} + \begin{bmatrix} K_\infty & \tilde{K}_\infty \\ \tilde{K}_\infty & K_\infty \end{bmatrix} =: \mathbb{P} + \mathbb{L}.
$$

- The system of integral equations becomes $(\mathbb{P}+\mathbb{L})\varphi = \mathbf{f}$.

Resonant Effect I: Resonance Condition

- Look for k such that $(\mathbb{P}+\mathbb{L})\varphi = 0$ attains non-trivial solutions.
- The operator equation reduces to

$$(\mathbb{M}+\mathbb{I})\begin{bmatrix}\langle\varphi,\mathbf{e}_1\rangle\\ \langle\varphi,\mathbf{e}_2\rangle\end{bmatrix} = 0,$$

where $\mathbf{e}_1 = [1,0]^T$ and $\mathbf{e}_2 = [0,1]^T$, and the matrix

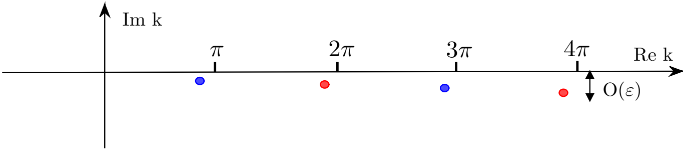
$$\mathbb{M} = \left(\beta\mathbb{I}+\tilde{\beta}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)\cdot\begin{bmatrix}\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\\ \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle & \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle\end{bmatrix}$$

- The eigenvalues of $\mathbb{M}+\mathbb{I}$ are given by

$$\lambda_1(k,\varepsilon) = 1+(\beta(k,\varepsilon)+\tilde{\beta}(k,\varepsilon))\left(\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle+\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\right),$$
$$\lambda_2(k,\varepsilon) = 1+(\beta(k,\varepsilon)-\tilde{\beta}(k,\varepsilon))\left(\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle-\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\right).$$

Resonance condition

The resonances are the roots of $\lambda_1(k,\varepsilon) = 0$ or $\lambda_2(k,\varepsilon) = 0$.



Theorem

The following asymptotic expansions hold for the resonances of the scattering problem:

$$k_{m,1} = (2m-1)\pi + 2(2m-1)\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}(2\ln 2 + \ln((2m-1)\pi) + \gamma_0)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon),$$

$$k_{m,2} = 2m\pi + 4m\pi\left[\frac{1}{\pi}\varepsilon\ln\varepsilon + \left(\frac{1}{\alpha} + \frac{1}{\pi}(2\ln 2 + \ln(2m\pi) + \gamma_0)\right)\varepsilon\right] + O(\varepsilon^2\ln^2\varepsilon),$$

for $m = 1, 2, 3, \cdots$, and $m\varepsilon \ll 1$. Here $\alpha = \langle K^{-1}1, 1\rangle$, $\gamma_0 = c_0 - \ln 2 - i\pi/2$, and c_0 is the Euler constant.

Remark The imaginary part of each resonance has an order of $O(\varepsilon)$.

Solution of the Operator Equation at Resonant Frequencies

- Solving the operator equation $(\mathbb{P}+\mathbb{L})\varphi = \mathbf{f}$ yields

$$\varphi \quad = \quad K^{-1}1 \cdot \left[d_1 \cdot O(k) \cdot \mathbf{e}_1 + \frac{\alpha}{\varepsilon \cdot \lambda_1}(\mathbf{e}_1 + \mathbf{e}_2) + \frac{\alpha}{\varepsilon \cdot \lambda_2}(\mathbf{e}_1 - \mathbf{e}_2) \right] + \text{H.O.T},$$

  where $\alpha = \langle K^{-1}1, 1 \rangle$.

- Away from the resonant frequencies, $\lambda_1 \sim O(1/\varepsilon)$, $\lambda_2 \sim O(1/\varepsilon)$, and consequently $\varphi \sim O(1)$.

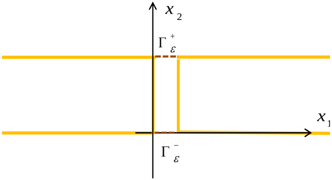
Solution at resonant frequencies

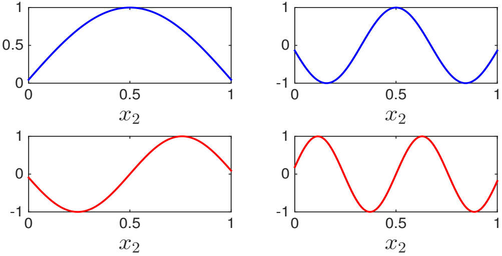
At the odd and even resonant frequencies $k = \Re k_{m,1}$ and $k = \Re k_{m,2}$,

$$\lambda_1 = -\frac{i\alpha}{2} + O(\varepsilon \ln^2 \varepsilon), \quad \lambda_2 = -\frac{i\alpha}{2} + O(\varepsilon \ln^2 \varepsilon),$$

and

$$\varphi \sim O(1/\varepsilon).$$





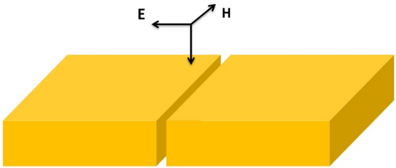
The wave field inside the slit adopts the following expansion at the odd and even resonances respectively:

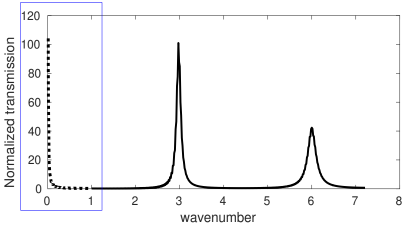
$$u_\varepsilon(x) = \frac{1}{\varepsilon} \cdot \frac{2i}{k\sin(k/2)} \cdot \cos(k(x_2 - 1/2)) + O(\ln^2 \varepsilon)$$

and

$$u_\varepsilon(x) = -\frac{1}{\varepsilon} \cdot \frac{2i}{k\cos(k/2)} \cdot \sin(k(x_2 - 1/2)) + O(\ln^2 \varepsilon).$$

Non-resonant Enhancement at Low Frequencies I





- Expand the wave field in the slit as the sum of wave-guide modes:

$$u_\varepsilon(x) = a_0 \cos kx_2 + b_0 \cos k(1-x_2) + \sum_{m\geq 1} \left[a_m \exp\left(-k_2^{(m)} x_2\right) + b_m \exp\left(-k_2^{(m)}(1-x_2)\right)\right] \cos\frac{m\pi x_1}{\varepsilon},$$

where $k_2^{(m)} = \sqrt{(m\pi/\varepsilon)^2 - k^2}$.

Lemma

$$a_0 = \frac{1}{k\sin k}\left[\alpha + O(k\varepsilon)\right]\cdot\left(\frac{1}{\varepsilon\cdot\lambda_1} + \frac{1}{\varepsilon\cdot\lambda_2}\right), \quad b_0 = \frac{1}{k\sin k}\left[\alpha + O(k\varepsilon)\right]\cdot\left(\frac{1}{\varepsilon\cdot\lambda_1} - \frac{1}{\varepsilon\cdot\lambda_2}\right),$$

$$\sqrt{m}\,|a_m| \leq C, \quad \sqrt{m}\,|b_m| \leq C, \quad \text{for } m \geq 1,$$

Non-resonant Enhancement at Low Frequencies II: In the Slit

- If $k \ll 1$ and $\varepsilon \ll 1$,

$$u_\varepsilon(x) = \left(\frac{\alpha}{\varepsilon\cdot\lambda_1} + \frac{\alpha}{\varepsilon\cdot\lambda_2}\right)\cdot\frac{\cos(kx_2)}{k\sin k} + \left(\frac{\alpha}{\varepsilon\cdot\lambda_1} - \frac{\alpha}{\varepsilon\cdot\lambda_2}\right)\cdot\frac{\cos(k(1-x_2))}{k\sin k} + H.O.T$$

$$= 2x_2 + O(k^2) + O(\varepsilon\ln(k\varepsilon)).$$ (The leading-order term has a slope of 2)

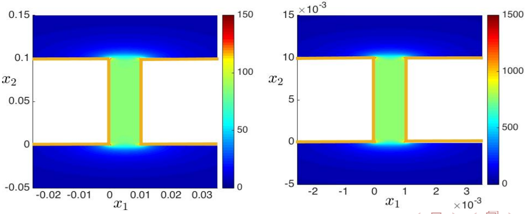
- By Ampere's law, the electric field $E_\varepsilon = [E_{\varepsilon,1}, E_{\varepsilon,2}, 0]$, where

$$E_{\varepsilon,1} = \frac{2i}{\sqrt{\tau_0/\mu_0}}\cdot\frac{1}{k} + O(\varepsilon\ln(k\varepsilon)/k) + O(k) \quad \text{and} \quad E_{\varepsilon,2} \sim O(\exp(-1/\varepsilon)/k).$$

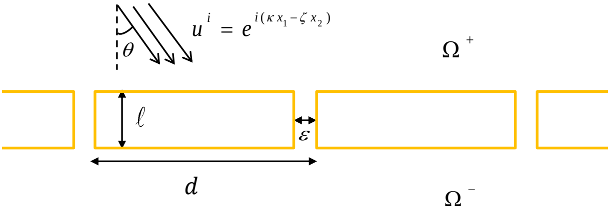
Theorem

No significant magnetic field enhancement is gained. However, the electric field $|E_\varepsilon| \sim O(1/k)$ or $|E_\varepsilon| \sim O(1/(k\ell))$ if $\ell \neq 1$.

$|E_\varepsilon|/|E^{inc}|$ for $k = 0.1$. Left: $\ell = 0.1$, $\varepsilon = 0.01$; Right: $\ell = 0.01$, and $\varepsilon = 0.001$.



Scattering by A Periodic Array of PEC Slits



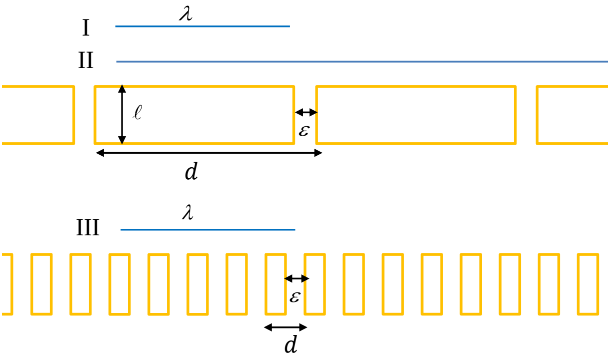
- A periodic array of slits: $S_\varepsilon = \bigcup_{n=-\infty}^{\infty} \left(S_\varepsilon^{(0)} + nd\right)$.
- The scattering problem: $\Delta u_\varepsilon + k^2 u_\varepsilon = 0$ in Ω_ε and $\partial_\nu u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$.
- Look for quasi-periodic solutions such that $u_\varepsilon(x_1 + d, x_2) = e^{i\kappa d} u_\varepsilon(x_1, x_2)$.
- Outgoing radiation condition: the scattered field

$$u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,\pm} e^{i\kappa_n x_1 \pm i\zeta_n x_2} \quad \text{in } \Omega^\pm,$$

where

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2 - \kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

Three Configurations of Periodic Slits



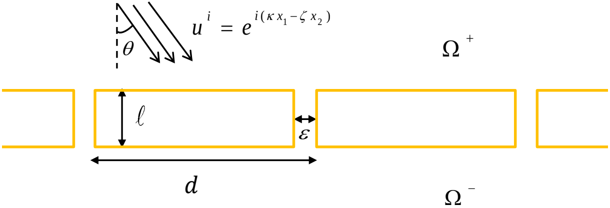
- Normalization: $\ell = 1$.
- Three configurations of periodic slits:

(I) $\varepsilon \ll d \sim \lambda \sim O(1)$: diffraction regime.

(II) $\varepsilon \ll d \ll \lambda$: homogenization regime I

(III) $\varepsilon \sim d \ll \lambda \sim O(1)$: homogenization regime II

Diffraction Regime: $\varepsilon \ll d \sim \lambda$

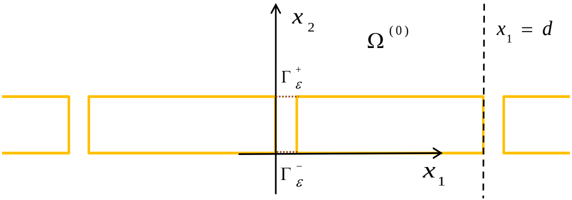


- Reduce to the first Brillouin zone: $\kappa \in (-\pi/d, \pi/d]$.
- Exterior Green's function in $\Omega^{\pm}$: $g_\sharp^e(x,y) = g_\sharp^d(x,y) + g_\sharp^d(x',y)$, where

$$g_\sharp^d(x,y) = -\frac{i}{2d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} e^{i\kappa_n(x_1-y_1)+i\zeta_n(k)|x_2-y_2|},$$

and

$$\kappa_n = \kappa + \frac{2\pi n}{d} \quad \text{and} \quad \zeta_n(k) = \begin{cases} \sqrt{k^2-\kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2-k^2}, & |\kappa_n| > k. \end{cases}$$



- Integral equation formulation over one reference period:

$$\begin{cases} \displaystyle\int_{\Gamma_\varepsilon^+} g_\sharp^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = -(u^i+u^r), & \text{on } \Gamma_\varepsilon^+, \\ \displaystyle\int_{\Gamma_\varepsilon^-} g_\sharp^e(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y + \int_{\Gamma_\varepsilon^+\cup\Gamma_\varepsilon^-} g_\varepsilon^i(x,y)\frac{\partial u_\varepsilon}{\partial \nu}ds_y = 0, & \text{on } \Gamma_\varepsilon^-. \end{cases}$$

- Boundary integral equation after scaling:

$$\begin{bmatrix} T_\sharp^e + T^i & \tilde{T}^i \\ \tilde{T}^i & T_\sharp^e + T^i \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} f/\varepsilon \\ 0 \end{bmatrix}.$$

$T_\sharp^e$ is the integral operator with kernel $G_{\sharp,\varepsilon}^e$:

$$G_{\sharp,\varepsilon}^e(X,Y) = \frac{1}{\pi}\left(\ln\varepsilon + \ln 2 + \ln\frac{\pi}{d}\right) + \frac{1}{2\pi}\sum_{n\neq 0}\frac{1}{|n|} - \frac{i}{d}\sum_{n=-\infty}^{\infty}\frac{1}{\zeta_n(k)} + \frac{1}{\pi}\ln|X-Y| + O(\varepsilon|X-Y|).$$

- Asymptotics of integral operators and the resonance condition can be obtained!

Diffraction Regime: Rayleigh Anomaly Frequencies

- Rayleigh anomaly frequencies: $k = \kappa_n = \kappa + 2\pi n/d$ or $\zeta_n = 0$ for some n.

  Note that the scattered field

$$u_\varepsilon^s(x_1, x_2) = \sum_{n=-\infty}^{\infty} u_n^{s,\pm} e^{i\kappa_n x_1 \pm i\zeta_n x_2}, \quad \zeta_n(k) = \begin{cases} \sqrt{k^2 - \kappa_n^2}, & |\kappa_n| \le k, \\ i\sqrt{\kappa_n^2 - k^2}, & |\kappa_n| > k. \end{cases}$$

- Resonances away from the Rayleigh anomaly frequencies: consider the domain

$$D_{\kappa,\delta,M} := \mathbf{C} \backslash B_{\kappa,\delta} \cap \{z \mid |z| \le M\}, \quad \text{where } B_{\kappa,\delta} := \bigcup_{n=-\infty}^{\infty} B_\delta(\kappa + 2\pi n/d).$$

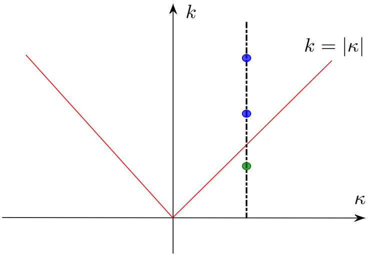
Theorem

For each $\kappa \in (-\pi/d, \pi/d]$, if $m\pi \in D_{\kappa,\delta,M}$, there exists a resonance or an eigenvalue k_m in the neighborhood of $m\pi$.

- If $m\pi > |\kappa|$, k_m is a resonance. Otherwise, k_m is an eigenvalue.
- The following asymptotic expansion holds for k_m if $m\varepsilon \ll 1$:

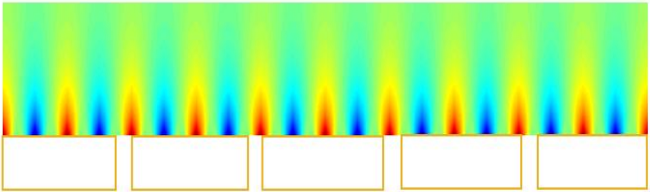
$$k_m = m\pi + 2m\pi \left[\frac{1}{\pi} \varepsilon \ln \varepsilon + \left(\frac{1}{\alpha} + \gamma(m\pi, \kappa, d) \right) \varepsilon \right] + O(\varepsilon^2 \ln^2 \varepsilon),$$

Here $\alpha = \langle K^{-1}1, 1\rangle$, $\gamma(k,\kappa,d) = \frac{1}{\pi}\left(3\ln 2 + \ln\frac{\pi}{d}\right) + \left(\frac{1}{2\pi} \sum_{n \neq 0} \frac{1}{|n|} - \frac{i}{d} \sum_{n=-\infty}^{\infty} \frac{1}{\zeta_n(k)} \right)$.



- Im $\gamma(m\pi, \kappa, d) = -\frac{1}{d} \sum_{|\kappa_n| < m\pi} \frac{1}{\zeta_n(m\pi)} < 0$ if $m\pi > |\kappa|$, and the resonance has an imaginary part of $O(\varepsilon)$.
- Im $\gamma(m\pi, \kappa, d) = 0$ if $m\pi < |\kappa|$.
- The eigenvalue occurs only if $d < 1$.
- The eigenmode u_ε^s is a surface bound state (decaying exponetial away from the grating surface).

Surface Bound State



Diffraction regime: Field Enhancement at Resonant Frequencies I

- Solving the operator equation $(\mathbb{P}+\mathbb{L})\varphi = \mathbf{f}$ yields

$$\varphi = K^{-1}1\cdot\left[\kappa\cdot O(1)\cdot\mathbf{e}_1 + \frac{\alpha}{\varepsilon\cdot\lambda_1}(\mathbf{e}_1+\mathbf{e}_2) + \frac{\alpha}{\varepsilon\cdot\lambda_2}(\mathbf{e}_1-\mathbf{e}_2)\right] + \text{H.O.T},$$

and

$$\lambda_1(k;\kappa,d,\varepsilon) = 1+(\beta+\tilde{\beta})\left(\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle + \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\right),$$
$$\lambda_2(k;\kappa,d,\varepsilon) = 1+(\beta-\tilde{\beta})\left(\langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_1\rangle - \langle\mathbb{L}^{-1}\mathbf{e}_1,\mathbf{e}_2\rangle\right).$$

- Away from the resonant frequencies, $\lambda_1 \sim O(1/\varepsilon)$, $\lambda_2 \sim O(1/\varepsilon)$, and consequently $\varphi \sim O(1)$.

Scattering solution at resonant frequencies

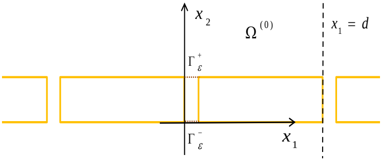
At the odd and even resonant frequencies $k = \text{Re}\, k_{m,1}$ and $k = \text{Re}\, k_{m,2}$,

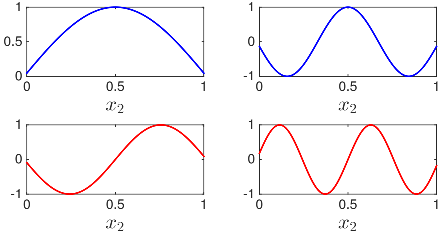
$$\lambda_1 = i\cdot\text{Im}\,\gamma(m\pi,\kappa,d)\cdot\alpha + O(\varepsilon\ln^2\varepsilon), \quad \lambda_2 = i\cdot\text{Im}\,\gamma(m\pi,\kappa,d)\cdot\alpha + O(\varepsilon\ln^2\varepsilon),$$

and

$$\varphi \sim O(1/\varepsilon).$$

Field Enhancement at Resonant Frequencies II





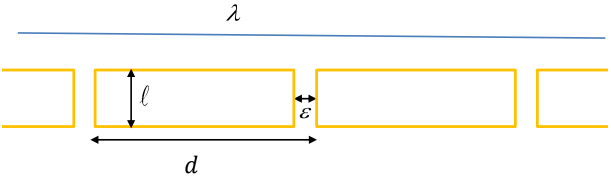
In the slit $S_\varepsilon^{(0)}$

The wave field adopts the following expansion at the odd and even resonances respectively:

$$u_\varepsilon(x) = \frac{1}{\varepsilon} \cdot \frac{i}{\mathrm{Im}\,\gamma(m\pi,\kappa,d) \cdot k \sin(k/2)} \cdot \cos(k(x_2 - 1/2)) + O(\ln^2 \varepsilon)$$

and

$$u_\varepsilon(x) = -\frac{1}{\varepsilon} \cdot \frac{i}{\mathrm{Im}\,\gamma(m\pi,\kappa,d) \cdot k \cos(k/2)} \cdot \sin(k(x_2 - 1/2)) + O(\ln^2 \varepsilon).$$



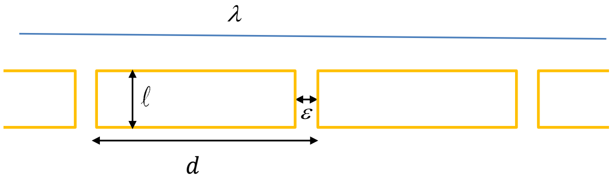
- No scattering resonance or eigenvalue exists if $k \ll 1$ (or $\lambda \gg 1$).

- If $\varepsilon \ll 1$ and $k = \varepsilon^{\sigma}$, in the reference slit,

$$u_\varepsilon(x) = \left(\frac{\alpha}{\varepsilon \cdot \lambda_1} + \frac{\alpha}{\varepsilon \cdot \lambda_2}\right) \cdot \frac{\cos(kx_2)}{k \sin k} + \left(\frac{\alpha}{\varepsilon \cdot \lambda_1} - \frac{\alpha}{\varepsilon \cdot \lambda_2}\right) \cdot \frac{\cos(k(1-x_2))}{k \sin k} + H.O.T$$
$$= \begin{cases} 2x_2 + O(\varepsilon^{2\sigma}) + O(\varepsilon^{1-\sigma}) & \text{if } 0 < \sigma < 1, \\ 1 + id \cdot \cos\theta \left(2x_2 - 1\right) \varepsilon^{\sigma-1} + O(\varepsilon^{\sigma+1}) + O(\varepsilon^{2(\sigma-1)}) & \text{if } \sigma > 1, \end{cases}$$

- No magnetic enhancement is gained. However, the leading-order term has a slope of 2 and $O(\varepsilon^{\sigma-1})$ respectively .

Homogenization Regime I: Non-resonant Field Enhancement

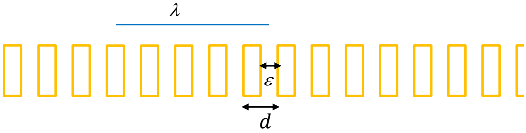


Electric field enhancement

If $\varepsilon \ll 1$ and $k = \varepsilon^\sigma$, then $E_\varepsilon = [E_{\varepsilon,1}, E_{\varepsilon,2}, 0]$ in the reference slit, where

$$E_{\varepsilon,1} = \begin{cases} \dfrac{2i}{\sqrt{\tau_0/\mu_0}} \cdot \dfrac{1}{\varepsilon^\sigma} + H.O.T & \text{if } 0 < \sigma < 1, \\ \dfrac{d\cos\theta}{\sqrt{\tau_0/\mu_0}} \cdot \dfrac{1}{\varepsilon} + H.O.T & \text{if } \sigma > 1, \end{cases} \quad \text{and} \quad E_{\varepsilon,2} \sim O(e^{-1/\varepsilon}).$$

Homogenization Regime II: $\varepsilon \sim d \ll \lambda$



- $\eta := \varepsilon/d$, where $0 < \eta < 1$.

- Asymptotic expansion of the scattering solution can be obtained, using the expansion for the periodic Green's function:

$$G_\varepsilon^e(X,Y) = \frac{1}{\pi}\ln 2 - \frac{i\eta}{\zeta\varepsilon} + \frac{1}{\pi}\ln|\sin(\pi\eta(X-Y))| + \frac{\kappa\eta}{\zeta}(X-Y) + O(\varepsilon),$$

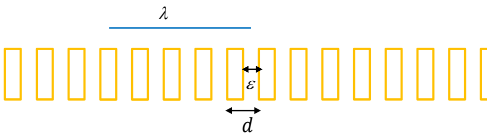
 where $\kappa^2 + \zeta^2 = k^2$.

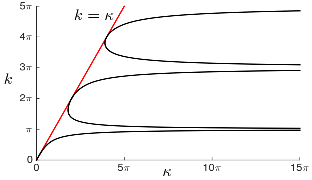
- Solving the homogeneous scattering problem leads to

$$\hat{\lambda}_1(k,\varepsilon) := 1 + \left[\left(\frac{\cot k}{k} + \frac{1}{k\sin k} - \frac{i\eta}{\zeta}\right)\frac{1}{\varepsilon} + \frac{3\ln 2}{\pi}\right]\left(\langle \hat{\mathbb{L}}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle + \langle \hat{\mathbb{L}}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right) = 0,$$

$$\hat{\lambda}_2(k,\varepsilon) := 1 + \left[\left(\frac{\cot k}{k} - \frac{1}{k\sin k} - \frac{i\eta}{\zeta}\right)\frac{1}{\varepsilon} + \frac{3\ln 2}{\pi}\right]\left(\langle \hat{\mathbb{L}}^{-1}\mathbf{e}_1, \mathbf{e}_1\rangle - \langle \hat{\mathbb{L}}^{-1}\mathbf{e}_1, \mathbf{e}_2\rangle\right) = 0.$$

Homogenization Regime II: “Surface Plasmon"

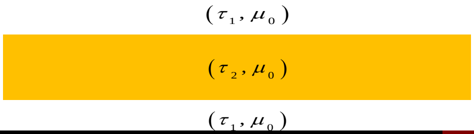


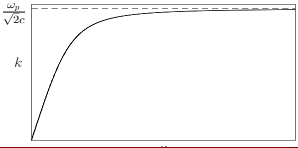


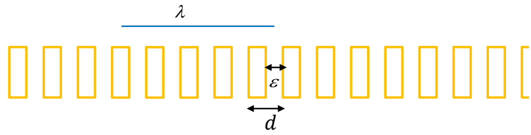
Theorem

There exist two groups of dispersion relations satisfying $|\kappa| > k$, and their leading orders are: $\kappa = k\sqrt{1+\eta^2\left(\frac{\sin k}{\cos k \pm 1}\right)^2}, \quad \eta = \varepsilon/d.$

- The associated eigenmodes u_ε^s are surface bound states.
- The dispersion relations and surface bound states resemble the ones for surface plasmon polaritons in the dielectric-metal configuration.





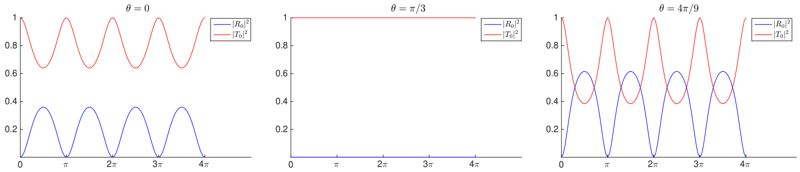


- Scattering by an incident plane wave $u^i = e^{i(\kappa x_1 - \zeta(x_2-1))}$, where $\kappa = k\sin\theta$, $\zeta = k\cos\theta$, and $|\kappa| < k$.

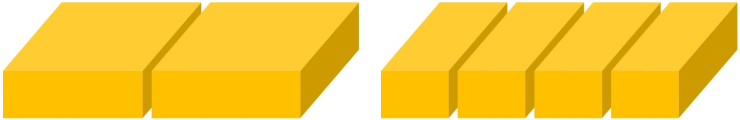
- The leading orders of the reflection and transmission coefficients are

$$R_0 = \frac{i\tan k\cdot(\eta^2-\cos^2\theta)}{-i\tan k\cdot(\eta^2+\cos^2\theta))+2\eta\cos\theta}, \quad T_0 = \frac{2\cos\theta\cdot\eta}{-i\sin k\cdot(\eta^2+\cos^2\theta)+2\cos\theta\cdot\eta\cos k}.$$

- Total transmission is achieved when $k = m\pi$ (Fabry-Perot resonance), and all frequencies for a special incident angle θ such that $\cos\theta = \eta$ (Brewster angle).



A Quick Summary

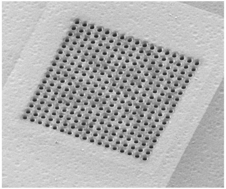


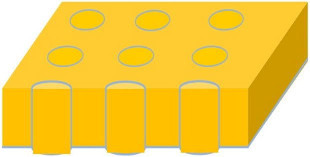
Field enhancement for PEC metals:

- Single slit: resonant and non-resonant enhancement effects.
- An array of slits: resonant and non-resonant enhancement effects, surface bound states, “surface plasmon", and total transmission.
- Asymptotics of resonances/eigenvalues are derived, and the enhanced wave modes are characterized.

Outlook

1. Field enhancement for “real" metals:
   - Shift of resonances
   - Field enhancement at the presence of metal loss
   - Plasmonic effect
2. 3D subwavelength structures: quantitative analysis





Thank you for your attention!